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Consensus control of second-order delayed multiagent systems with intrinsic dynamics and measurement noises

Xiaofeng Zong[1,2] | Tao Li[3] | Ji-Feng Zhang[4,5]

[1]School of Automation, China University of Geosciences, Wuhan 430074, China

[2]Hubei Key Laboratory of Advanced Control and Intelligent Automation for Complex Systems, Wuhan 430074, China

[3]Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, School of Mathematical Sciences, East China Normal University, Shanghai 200241, China

[4]Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

[5]School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

Correspondence
Tao Li, School of Mathematical Sciences, East China Normal University, Shanghai 200241, China.
Email: tli@math.ecnu.edu.cn

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Summary

This paper studies the consensus control of second-order multiagent systems with intrinsic dynamics based on delayed and noisy measurements, where the delays in the position and velocity measurements are allowed to be different. The nonlinear and linear intrinsic dynamics are considered, respectively. For the case with nonlinear dynamics, mean square and almost sure consensus conditions are established by applying the degenerate Lyapunov functional and stochastic stability theorems. For the delay-free case with linear dynamics, appropriate Lyapunov functions are established to get some simple sufficient conditions for mean square and almost sure consensus, and necessary conditions for mean square consensus. It is shown that, with respect to the weighted average type control protocols, second-order multiagent systems are kept mean square consentable under multiplicative measurement noises alone or intrinsic dynamics alone, but may become unconsentable due to the coexistence of multiplicative noises and intrinsic dynamics. These results are further extended to leader-following multiagent systems.

KEYWORDS

consensus, delay, intrinsic dynamics, noise, multiagent systems

1 | INTRODUCTION

Recently, there has been considerable attention paid to the study of multiagent systems due to their flexibility and intelligence in solving distributed problems. Potential applications of multiagent systems are numerous, including societies, markets, satellite communications, and biological systems.[1-5] One of the important research interests in multiagent systems is the consensus control, which focuses on the design of distributed protocols to guarantee certain agreement or synchronization. That is, the states (or partial states) of all agents can reach the same value or function by local information exchange.[6-9]

Note that measurement noises often exist in data transmission channels.[10,11] Hence, multiagent systems subjected to measurement noises have to be taken into account. In the literature, both the additive and multiplicative noises were investigated in the consensus control of multiagent systems. For the case with additive noises, the stochastic approximation method was applied for discrete-time model and continuous-time model to examine mean square and almost sure consensus,[12-18] where the consensus conditions related to control gain functions were found. For the case with multiplicative noises, stochastic stability and martingale convergence theorems were applied to establish necessary conditions and sufficient conditions for mean square and almost sure consensus of continuous-time models in the works of Ni and Li[19] and Li et al,[20] and sufficient consensus conditions of discrete-time models in other works.[21-23] When the measurement delay also appears in multiagent systems, Liu et al[24] got sufficient conditions for mean square average consensus under additive measurement noises by developing the generalized Gronwall-Bellman-Halanay type inequality, and our previous work[25] obtained necessary and sufficient conditions for mean square and almost sure weak and strong consensus under multiplicative measurement noises by using degenerate Lyapunov functionals.

The consensus analysis aforementioned focused on the first-order multiagent systems. In practical applications, many mobile agents are determined by both position and velocity states, such as torque motors and gas jets, which are adjusted for their desired motion directly by the acceleration rather than the speed. Therefore, more and more attention has been paid to consensus problems of second-order multiagent systems. In the work of Yu et al,[26] the necessary and sufficient consensus conditions were obtained for the delayed and delay-free cases, respectively. In the work of Carli et al,[27] a second-order consensus algorithm was proposed for a family of nonidentical double integrators. Eichler and Werner[28] studied the constrained min-max problem, which optimizes the consensus speed subject to a lower bound on damping. Li and Chen[29] applied Riccati equations to establish mean square consensus of linear multiagent systems. Taking the measurement delay and multiplicative measurement noises into consideration, the second-order consensus conditions were obtained in the works of Zong et al.[30,31] Besides the second-order dynamic property, mobile agents may be governed by more complicated intrinsic dynamics.[32] With the interaction among neighboring agents, intrinsic nonlinear or linear dynamics will determinate the agents' behavior, such as the well-known Kuramoto oscillator.[33] Although some publications considered the second-order multiagent consensus with intrinsic dynamics,[34-39] little is known about the joint effects of measurement delays, noises, and intrinsic dynamics on multiagent consensus.

In this paper, we study the second-order consensus problem by analyzing the joint effects of measurement delays, multiplicative noises, and intrinsic dynamics on multiagent consensus. Here, the protocol is based on the relative position and velocity measurements according to the network structure, and the measurement delays in the relative position and velocity measurements are allowed to be different. Compared with the deterministic case, the consensus conditions of these models are difficult to be obtained in present of multiplicative noises. Recently, we have successfully established consensus conditions of linear second-order systems with measurement delays and multiplicative noises.[30] However, it cannot be applied to nonlinear second-order models, since the stochastic stability theorem developed in the work of Zong et al[30] is based on the stochastic delay equation with the linear drift. Moreover, delays appear not only in agents' external measurements but also in their intrinsic dynamics. This will add new difficulties in constructing the Lyapunov functional and finding consensus conditions, since it produces the pure delayed second-order stochastic system, whose stochastic stability criterion has not been well established. To this end, first, we make the most of the topology structure and convert the consensus problem into the stochastic stability issue of complicated stochastic systems, and then apply the stochastic stability theorem related to degenerate Lyapunov functionals to examine the consensus conditions in the senses of mean square and probability one.

The main contributions and findings are listed as follows.

- For the leader-free multiagent systems, explicit consensus conditions for nonlinear and linear second-order systems are established, respectively.
  - For the case with nonlinear dynamics and delays, we develop sufficient conditions for mean square and almost sure consensus and prove that, if the delays and noise intensities in the measurements and the Lipschitz constants of the nonlinear dynamics are small enough, then the closed-loop system achieves mean square and almost sure consensus under appropriate control gains.
  - For the case with linear dynamics, we obtain some necessary conditions and sufficient conditions for mean square consensus by choosing appropriate Lyapunov functions. It is revealed that, with respect to weighted average type

control protocols based on local information, second-order multiagent systems are kept mean square and almost surely consentable under multiplicative noises alone or intrinsic dynamics alone. However, the co-occurrence of multiplicative noises and intrinsic dynamics may weaken and destroy the mean square consentability, ie, the weighted average type control protocols based on the local information may not exist to guarantee the mean square consensus if intrinsic dynamics and multiplicative measurement noises appear simultaneously. This is different from the case without intrinsic dynamic.[31]

- For the leader-following multiagent systems, sufficient conditions for mean square and almost sure leader-following consensus are obtained under the assumption that the subgraph formed by the followers is undirected.

The organization of this paper is as follows. The consensus conditions of leader-free multiagent systems are addressed in Section 2, which contains two sections. In Section 2.1, sufficient conditions are established for mean square and almost sure consensus of second-order multiagent systems with nonlinear dynamics. In Section 2.2, some simple sufficient conditions for mean square and almost sure consensus, and necessary conditions for mean square consensus are obtained for the case with linear dynamics. In Section 3, consensus conditions for leader-following multiagent systems are obtained. The simulations are carried out in Section 4 to show the effectiveness of the theoretical method. The conclusion and future research are given in Section 5.

Notation. Throughout this paper, unless otherwise specified, we use the following notations. $\mathbf{1}_N$ denotes an N-dimensional column vector with all ones. $\eta_{N,i}$ denotes the N-dimensional column vector with the ith element being 1 and others being zero. $J_N = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$. I_N denotes the N-dimensional identity matrix. For a given matrix or vector A, its transpose is denoted by A^T, and its Euclidean norm is denoted by $|A|$. For two matrices A and B, $A \otimes B$ denotes their Kronecker product. For $a, b \in \mathbb{R}$, $a \vee b$ represents $\max\{a, b\}$ and $a \wedge b$ denotes $\min\{a, b\}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, namely, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. For a given random variable or vector X, the mathematical expectation of X is denoted by $\mathbb{E}X$. For a continuous martingale $M(t)$, its quadratic variation is denoted by $\langle M\rangle(t)$ (see the work of Revuz and Yor[40]). For the fixed $\tau > 0$, we use $C([-\tau, 0]; \mathbb{R}^n)$ to denote the space of all continuous $\mathbb{R}^n$-valued functions φ defined on $[-\tau, 0]$ with the norm $\|\varphi\|_C = \sup_{t\in[-\tau,0]}\|\varphi(t)\|$.

2 | LEADER-FREE MULTIAGENT SYSTEMS

Consider N agents distributed according to an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes with i representing the ith agent, $\mathcal{E}$ denotes the set of undirected edges, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with element $a_{ij} = 1$ or 0 indicating whether or not there is an information flow from agent j to agent i directly. N_i denotes the set of the node i's neighbors, ie, for $j \in N_i$, $a_{ij} = 1$. In addition, $\deg_i = \sum_{j=1}^N a_{ij}$ is called the degree of i. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(\deg_1, \dots, \deg_N)$. In this paper, $\mathcal{G}$ is assumed to be connected. In this case, $\mathcal{L}$ admits a zero eigenvalue, denoted by λ_1, and other eigenvalues $0 < \lambda_2 \leq \dots \leq \lambda_N$ are positive.

The dynamic of each agent is given by

$$\begin{cases} \dot{y}_i(t) = v_i(t), \\ \dot{v}_i(t) = f(y_i(t-\tau_y), v_i(t-\tau_v), t) + u_i(t), \end{cases} \tag{1}$$

where $y_i(t) \in \mathbb{R}^n$ and $v_i(t) \in \mathbb{R}^n$ denote the position and velocity of the ith agent, respectively. Here, τ_y and τ_v are delays in the intrinsic dynamic $f(y_i(t-\tau_y), v_i(t-\tau_v), t)$, $u_i(t)$ is the control input to be designed, and $f(\cdot, \cdot, t)$ is a nonlinear function. Let $y(t) = [y_1^T(t), \dots, y_N^T(t)]^T$, $v(t) = [v_1^T(t), \dots, v_N^T(t)]^T$, $x(t) = [y^T(t), v^T(t)]^T$, and $u(t) = [u_1^T(t), \dots, u_N^T(t)]^T$.

Remark 1. Before this paper, the consensus of nonlinear second-order multiagent systems has been investigated under general digraphs in the work of Yu et al[34] and switching graphs in other works.[37-39] These works are remarkable for deterministic models without time delay and measurement noise. In this paper, we take the time delays and measurement noises into consideration, where the time delays appear not only in the nonlinear dynamics but also in the measurements.

In this paper, we consider the following weighted average type control protocols:

$$u_i(t) = k_1 \sum_{j\in N_i} z_{yji}(t) + k_2 \sum_{j\in N_i} z_{vji}(t), \tag{2}$$

where $k_1 > 0, k_2 > 0, z_{yji}(t) = y_j(t-\tau_1) - y_i(t-\tau_1) + g_{1ji}(y_j(t-\tau_1) - y_i(t-\tau_1))\xi_{1ji}(t)$, and $z_{vji}(t) = v_j(t-\tau_2) - v_i(t-\tau_2) + g_{2ji}(v_j(t-\tau_2) - v_i(t-\tau_2))\xi_{2ji}(t)$ are the position and velocity measurements of agent i from its neighboring agent j, respectively, τ_1 and τ_2 are measurement delays, $\xi_{ji}(t) = [\xi_{1ji}(t), \xi_{2ji}(t)]^T$ is a two-dimensional noise, and $g_{1ji}(\cdot)$ and $g_{2ji}(\cdot)$ are intensity functions. We make the following hypotheses.

Hypothesis 1. *The noise process $\xi_{lji}(t) \in \mathbb{R}$ satisfies $\int_0^t \xi_{lji}(s)\mathrm{d}s = w_{lji}(t)$, $t \geq 0$, $l = 1, 2$, $i = 1, 2, \ldots, N$, $j \in N_i$, where $\{w_{lji}(t), l = 1, 2, i = 1, 2, \ldots, N, j \in N_i\}$ are independent scalar Brownian motions.*

Hypothesis 2. *For each (j, i), $g_{1ji}(0) = g_{2ji}(0) = 0$ and there exist positive constants $\bar{\sigma}_1$ and $\bar{\sigma}_2$ such that for all $y_1, y_2 \in \mathbb{R}^n$,*

$$\left|g_{1ji}(y_1) - g_{1ji}(y_2)\right| \leq \bar{\sigma}_1 |y_1 - y_2|$$

and

$$\left|g_{2ji}(y_1) - g_{2ji}(y_2)\right| \leq \bar{\sigma}_2 |y_1 - y_2|.$$

We assume that the initial data $x(t) = \varphi(t) \in C([-\tau, 0]; \mathbb{R}^{2n})$, $t \in [-\tau, 0]$, where $\tau = \tau_y \vee \tau_v \vee \tau_1 \vee \tau_2$. We aim to find the control gains k_1, k_2 such that the control $u(t)$ solves the consensus problems, where the definitions of the mean square and almost sure consensus are given as follows.

Definition 1. We say that the protocol $u(t)$ solves mean square (or almost sure) consensus if it makes the agents have the property that for any initial data φ and all distinct $i, j \in \mathcal{V}$, $\lim_{t\to\infty} \mathbb{E}|x_i(t) - x_j(t)|^2 = 0$ (or $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ almost surely.

Remark 2. The deterministic and ideal measurements have been examined in the works of Olfati-Saber and Murray[41] and Yu et al[26] for first-order model and second-order model, respectively. In this work, the measurements $z_{yji}(t)$ and $z_{vji}(t)$ can be considered as the joint impact of measurement delays and multiplicative noises on the ideal measurements $y_j(t) - y_i(t)$ and $v_j(t) - v_i(t)$ and were detailed in our recent works[25,30] for linear multiagent systems.

2.1 | Second-order consensus with nonlinear intrinsic dynamics

We first investigate the consensus conditions of multiagent systems with nonlinear dynamics. We assume that the function $f(y, v, t)$ satisfies the following hypothesis.

Hypothesis 3. *We assume that $f(0, 0, t) \equiv 0$ and there exist positive constants q_1 and q_2 (Lipschitz constants) such that*

$$|f(y, v, t) - f(\bar{y}, \bar{v}, t)| \leq q_1|y - \bar{y}| + q_2|v - \bar{v}|$$

for all $y, v, \bar{y}, \bar{v} \in \mathbb{R}^n$.

We will use the following notations. Define the unitary matrix $T_{\mathcal{L}} = [\frac{\mathbf{1}_N}{\sqrt{N}}, \phi_2, \ldots, \phi_N]$, where ϕ_i is the unit eigenvector of $\mathcal{L}$ associated with the eigenvalue $\lambda_i = \lambda_i(\mathcal{L})$, ie, $\phi_i^T \mathcal{L} = \lambda_i \phi_i^T$, $|\phi_i| = 1$, $i = 2, \ldots, N$. Denote $\phi = [\phi_2, \ldots, \phi_N]$ and $\Lambda = \mathrm{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N)$. Denote $\delta_y(t) = [(I_N - J_N) \otimes I_n] y(t)$, and $\delta_y(t) = [\delta_{y1}^T(t), \ldots, \delta_{yN}^T(t)]^T$, where $\delta_{yi} \in \mathbb{R}^n$, $i = 1, \ldots N$. Let $\delta_y(t) = (T_{\mathcal{L}} \otimes I_n)\widetilde{\delta}_y(t)$ and $\widetilde{\delta}_y(t) = [\widetilde{\delta}_{y1}^T(t), \ldots, \widetilde{\delta}_{yN}^T(t)]^T$, and then it can be verified that $\widetilde{\delta}_{y1}(t) = \frac{1}{\sqrt{N}}(\mathbf{1}_N^T \otimes I_n)[(I_n - J_n) \otimes I_n] y(t) \equiv 0$ since $\mathbf{1}_N^T(I_n - J_n) = 0$. Denote $\overline{\delta}_y(t) = [\widetilde{\delta}_{y2}^T(t), \ldots, \widetilde{\delta}_{yN}^T(t)]^T$. Similarly, we can define $\delta_v(t)$, $\widetilde{\delta}_v(t)$, and $\overline{\delta}_v(t)$. Denote $\overline{\delta}(t) = [\overline{\delta}_y^T(t), \overline{\delta}_v^T(t)]^T$.

Theorem 1. *Suppose that Hypotheses 1, 2, and 3 hold. If the control gains k_1 and k_2 satisfy*

$$2k_1\lambda_2 > 2q_1 + q_2 \tag{3}$$

and

$$\rho(k_1) < \varrho(k_2), \tag{4}$$

where $\rho(k_1) = \frac{(2+q_2+k_1\lambda_N(\tau_1+\tau_2)^2)\lambda_2}{2k_1\lambda_2 - 2q_1 - q_2}[((3 + q_1 + q_2)\tau_1 + 2\tau_2)k_1^2\lambda_N + (\tau_1 + \tau_2 + 1)q_1\lambda_2^{-1} + 2k_1^2\frac{N-1}{N}\bar{\sigma}_1^2] + \lambda_2 k_1(\tau_1 + \tau_2) + q_1 + q_2(\tau_1 + \tau_2 + 2)$, $\varrho(k_2) = 2\lambda_2 k_2(1 - k_2\frac{N-1}{N}\bar{\sigma}_2^2) - \lambda_2[(q_1 + q_2 + 2)\tau_2 + \tau_1]k_2^2\lambda_N$, then the protocol (2) solves mean square and almost sure consensus.

Proof. It is enough to prove that the protocol (2) with k_1 and k_2 satisfying (3) and (4) can solve mean square and almost sure consensus. The proof is divided into the following four steps.

Step 1: Transformation of consensus problem into stability problem. We transform the consensus problem of multiagent system into the stability of a stochastic delayed equation. Let $F(t) = [f^T(y_1(t-\tau_y), v_1(t-\tau_v), t), \dots, f^T(y_N(t-\tau_y), v_N(t-\tau_v), t)]^T$. Substituting the protocol (2) into the system (1) and using Hypothesis 1 yield

$$dv(t) = F(t)dt - k_1(\mathcal{L} \otimes I_n)y(t-\tau_1)dt - k_2(\mathcal{L} \otimes I_n)v(t-\tau_2)dt + d\bar{M}(t),$$

where $\bar{M}(t) = k_1 \sum_{i,j=1}^N a_{ij} \int_0^t [\eta_{N,i} \otimes \tilde{g}_{1ji}(s-\tau_1)]\mathrm{d}w_{1ji}(s) + k_2 \sum_{i,j=1}^N a_{ij} \int_0^t [\eta_{N,i} \otimes \tilde{g}_{2ji}(s-\tau_2)]\mathrm{d}w_{2ji}(s)$, and $\tilde{g}_{1ji}(s) = g_{1ji}(\delta_{yj}(s) - \delta_{yi}(s)), \tilde{g}_{2ji}(s) = g_{2ji}(\delta_{vj}(s) - \delta_{vi}(s))$. Noting that $\delta_y(t) = [(I_N - J_N) \otimes I_n]y(t)$ and $\delta_v(t) = [(I_N - J_N) \otimes I_n]v(t)$, then we have from $(I_N - J_N)\mathcal{L} = \mathcal{L}(I_N - J_N)$ that

$$\begin{cases} d\delta_y(t) = \delta_v(t)dt \\ d\delta_v(t) = \bar{F}(t)dt - k_1(\mathcal{L} \otimes I_n)\delta_y(t-\tau_1)dt \\ \qquad\qquad -k_2(\mathcal{L} \otimes I_n)\delta_v(t-\tau_2)dt + d\bar{M}_0(t), \end{cases}$$

where $\bar{F}(t) = [(I_N - J_N) \otimes I_n]F(t)$ with $\bar{F}(t) = [\bar{f}_1^T(t), \dots, \bar{f}_N^T(t)]^T$ and $\bar{f}_i(t) = f(y_i(t-\tau_y), v_i(t-\tau_v), t) - \frac{1}{N}\sum_{j=1}^N f(y_j(t-\tau_y), v_j(t-\tau_v), t)$, $\bar{M}_0(t) = k_1 \sum_{i,j=1}^N a_{ij} \int_0^t [(I_N - J_N)\eta_{N,i} \otimes \tilde{g}_{1ji}(s-\tau_1)]\mathrm{d}w_{1ji}(s) + k_2 \sum_{i,j=1}^N a_{ij} \int_0^t [(I_N - J_N)\eta_{N,i} \otimes \tilde{g}_{2ji}(s-\tau_2)]\mathrm{d}w_{2ji}(s)$. Bear it in mind that $\widetilde{\delta}_y(t) = (T_{\mathcal{L}}^T \otimes I_n)\delta_y(t)$, $\widetilde{\delta}_v(t) = (T_{\mathcal{L}}^T \otimes I_n)\delta_v(t)$, $\widetilde{\delta}_{y1}(t) = 0$, and $\widetilde{\delta}_{v1}(t) = 0$. Then, we have the following delayed second-order stochastic system:

$$\begin{cases} d\bar{\delta}_y(t) = \bar{\delta}_v(t)dt \\ d\bar{\delta}_v(t) = (\phi^T \otimes I_n)\bar{F}(t)dt - k_1(\Lambda \otimes I_n)\bar{\delta}_y(t-\tau_1)dt \\ \qquad\qquad -k_2(\Lambda \otimes I_n)\bar{\delta}_v(t-\tau_2)dt + d\bar{M}_1(t) + d\bar{M}_2(t), \end{cases} \tag{5}$$

where $\bar{M}_1(t) = k_1 \sum_{i,j=1}^N a_{ij} \int_0^t [\bar{Q}(i) \otimes \tilde{g}_{1ji}(s-\tau_1)]\mathrm{d}w_{1ji}(s)$ and $\bar{M}_2(t) = k_2 \sum_{i,j=1}^N a_{ij} \int_0^t [\bar{Q}(i) \otimes \tilde{g}_{2ji}(s-\tau_2)]\mathrm{d}w_{2ji}(s)$ with $\bar{Q}(i) = \phi(I_N - J_N)\eta_{N,i}$. Let $L = L_0 + L_1 + L_2$ with $L_0 = \begin{bmatrix} 0 & I_{N-1} \\ 0 & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & 0 \\ -k_1\Lambda & 0 \end{bmatrix}$, and $L_2 = \begin{bmatrix} 0 & 0 \\ 0 & -k_2\Lambda \end{bmatrix}$. Then, we have

$$d\bar{\delta}(t) = \widetilde{F}(t)dt + (L_0 \otimes I_n)\bar{\delta}(t)dt + (L_1 \otimes I_n)\bar{\delta}(t-\tau_1)dt + (L_2 \otimes I_n)\bar{\delta}(t-\tau_2)dt + d\bar{M}_3(t) + d\bar{M}_4(t), \tag{6}$$

where $\widetilde{F}(t) = [0, (\phi^T \otimes I_n)\bar{F}^T(t)]^T$, $\bar{M}_3(t) = k_1 \sum_{i,j=1}^N a_{ij} \int_0^t B_{1ij}(s-\tau_1)\mathrm{d}w_{1ji}(s)$, $\bar{M}_4(t) = k_2 \sum_{i,j=1}^N a_{ij} \int_0^t B_{2ij}(s-\tau_2)\mathrm{d}w_{2ji}(s)$, $B_{1ij}(t) = [0, b_{1ij}^T(t)]^T$, $b_{1ij}(t) = \overline{Q}(i) \otimes \tilde{g}_{1ji}(t)$, $B_{2ij}(t) = [0, b_{2ij}^T(t)]^T$, and $b_{2ij}(t) = \bar{Q}(i) \otimes \tilde{g}_{2ji}(t)$. Note that $\delta_{yi}(t) = y_i - \sum_{k=1}^N y_k(t)/N$, $\delta_{vi}(t) = v_i - \sum_{k=1}^N v_k(t)/N$, $i = 1, \dots, N$. Hence, by the definition of $\bar{\delta}(t)$, we can see easily that the consensus problem is converted into the stability problem of (6).

Step 2: Construction of degenerate Lyapunov functional. To get the stability analysis of (6), we need to construct a degenerate Lyapunov functional. Let

$$P = \begin{bmatrix} \mu\Lambda & \theta I_{N-1} \\ \theta I_{N-1} & I_{N-1} \end{bmatrix} \tag{7}$$

with $\mu, \theta > 0$ to be designed. In fact, we need $\theta^2 < \mu\lambda_2$ to guarantee the positive definiteness of P. Let

$$z(t) = \bar{\delta}(t) + (L_1 \otimes I_n)\int_{t-\tau_1}^t \bar{\delta}(s)\mathrm{d}s + (L_2 \otimes I_n)\int_{t-\tau_2}^t \bar{\delta}(s)\mathrm{d}s. \tag{8}$$

We choose the degenerate Lyapunov functional

$$V(\bar{\delta}_t) = V_1(\bar{\delta}(t)) + V_2(t), \tag{9}$$

where

$$V_1(\bar{\delta}_t) = z(t)^T(P \otimes I_n)z(t) \tag{10}$$

and

$$V_2(\bar{\delta}_t) = \int_{-\tau_1}^{0}\left[\int_{t+s}^{t}\bar{\delta}^T(\theta)\left(L_1^TPL_1\otimes I_n\right)\bar{\delta}(\theta)\mathrm{d}\theta\right]\mathrm{d}s + \int_{-\tau_2}^{0}\left[\int_{t+s}^{t}\bar{\delta}^T(\theta)\left(L_2^TPL_2\otimes I_n\right)\bar{\delta}(\theta)\mathrm{d}\theta\right]\mathrm{d}s$$
$$+(\theta+1)\left[q_1\int_{t-\tau_y}^{t}|\bar{\delta}_y(s)|^2\mathrm{d}s + q_2\int_{t-\tau_v}^{t}|\bar{\delta}_v(s)|^2\mathrm{d}s\right] + k_1^2(q_1+q_2)\int_{-\tau_1}^{0}\int_{t+s}^{t}\bar{\delta}_y^T(u)(\Lambda^2\otimes I_n)\bar{\delta}_y(u)\mathrm{d}u\mathrm{d}s$$
$$+k_2^2(q_1+q_2)\int_{-\tau_2}^{0}\int_{t+s}^{t}\bar{\delta}_v^T(u)(\Lambda^2\otimes I_n)\bar{\delta}_v(u)\mathrm{d}u\mathrm{d}s + (\tau_1+\tau_2)(q_1\int_{t-\tau_y}^{t}|\bar{\delta}_y(s)|^2\mathrm{d}s + q_2\int_{t-\tau_v}^{t}|\bar{\delta}_v(s)|^2\mathrm{d}s)$$
$$+2\frac{N-1}{N}k_1^2\sigma_1^2\int_{t-\tau_1}^{t}\bar{\delta}_y^T(s)[\Lambda\otimes I_n]\bar{\delta}_y(s)\mathrm{d}s + 2\frac{N-1}{N}k_2^2\sigma_2^2\int_{t-\tau_2}^{t}\bar{\delta}_v^T(s)[\Lambda\otimes I_n]\bar{\delta}_v(s)\mathrm{d}s \tag{11}$$

with $\bar{\delta}_t = \{\bar{\delta}(t+\theta) : \theta\in[-\tau,0]\}$. In the following, we will give an estimation of the functional (9).

Step 3: Computation and estimation of Itô operator $\mathcal{L}V(\cdot)$. Computing the Itô operator $\mathcal{L}V_1(\cdot)$ on the solution path $\bar{\delta}(t)$ yields

$$\mathfrak{L}V_1(\bar{\delta}_t) = 2z^T(t)(P\otimes I_n)\left[\widetilde{F}(t) + (L\otimes I_n)\bar{\delta}(t)\right] + J_1(t)$$
$$= \bar{\delta}^T(t)\left[(L^TP+PL)\otimes I_n\right]\bar{\delta}(t) + \sum_{j=1}^{6}J_j(t),$$

where

$$J_1(t) = k_1^2\sum_{i,j=1}^{N}a_{ij}\left|b_{1ji}(t-\tau_1)\right|^2 + k_2^2\sum_{i,j=1}^{N}a_{ij}\left|b_{2ji}(t-\tau_2)\right|^2,$$
$$J_2(t) = 2\bar{\delta}^T(t)(P\otimes I_n)\widetilde{F}(t),$$
$$J_3(t) = 2\bar{\delta}^T(t)(L^TPL_1\otimes I_n)\int_{t-\tau_1}^{t}\bar{\delta}(s)\mathrm{d}s,$$
$$J_4(t) = 2\bar{\delta}^T(t)(L^TPL_2\otimes I_n)\int_{t-\tau_2}^{t}\bar{\delta}(s)\mathrm{d}s,$$
$$J_5(t) = 2\int_{t-\tau_1}^{t}\bar{\delta}^T(s)\mathrm{d}s\left(L_1^TP\otimes I_n\right)\widetilde{F}(t),$$
$$J_6(t) = 2\int_{t-\tau_2}^{t}\bar{\delta}^T(s)\mathrm{d}s\left(L_2^TP\otimes I_n\right)\widetilde{F}(t).$$

Note that $\phi\phi^T = I_N - J_N$ and $(I_N-J_N)^2 = (I_N-J_N)$, $\eta_{N,i}^T(I_N-J_N)\eta_{N,i} = \frac{N-1}{N}$. From the definition of $\bar{\delta}(t)$, Hypothesis 2, and $2\bar{\delta}_y^T(t)[\Lambda\otimes I_n]\bar{\delta}_y(t) = \sum_{i=1}^{N}\sum_{i=1}^{N}a_{ij}|\delta_{yj}(t)-\delta_{yi}(t)|^2$, we obtain

$$J_1(t) \le 2\frac{N-1}{N}k_1^2\sigma_1^2\bar{\delta}_y^T(t-\tau_1)[\Lambda\otimes I_n]\bar{\delta}_y(t-\tau_1)$$
$$+2\frac{N-1}{N}k_2^2\sigma_2^2\bar{\delta}_v^T(t-\tau_2)[\Lambda\otimes I_n]\bar{\delta}_v(t-\tau_2). \tag{12}$$

Let $\bar{y}(t) = \frac{1}{N}\sum_{j=1}^{N}y_j(t)$ and $\bar{v}(t) = \frac{1}{N}\sum_{j=1}^{N}v_j(t)$; then, $\delta_{yi}(t) = y_i(t)-\bar{y}(t)$ and $\delta_{vi}(t) = v_i(t)-\bar{v}(t)$. Note that $\bar{f}_i(t) =: p_{i1}(t)+p_{i2}(t)+p_{03}(t)$, where $p_{i1}(t) = f(y_i(t-\tau_y), v_i(t-\tau_v), t) - f(y_i(t-\tau_y), \bar{v}(t-\tau_v), t)$, $p_{i2}(t) = f(y_i(t-\tau_y), \bar{v}(t-\tau_v), t) - f(\bar{y}(t-\tau_y), \bar{v}(t-\tau_v), t)$, $p_{03}(t) = \frac{1}{N}\sum_{j=1}^{N}\Delta_j f(t)$, and $\Delta_j f(t) = f(\bar{y}(t-\tau_y), \bar{v}(t-\tau_v), t) - f(y_j(t-\tau_y), v_j(t-\tau_v), t)$. Then, $\bar{F}(t) = p_1(t) + p_2(t) + \mathbf{1}_N\otimes p_{03}(t)$, and we have

$$J_2(t) = 2\theta\bar{\delta}_y^T(t)(\phi^T\otimes I_n)\bar{F}(t) + 2\bar{\delta}_v^T(t)(\phi^T\otimes I_n)\bar{F}(t).$$

By Hypothesis 3, we obtain

$$2\delta_{yi}^T(t)p_{i1}(t) \le q_2\left(|\delta_{yi}(t)|^2 + |\delta_{vi}(t-\tau_v)|^2\right),$$

$$2\delta_{yi}^T(t)p_{i2}(t) \le q_1\left(|\delta_{yi}(t)|^2 + |\delta_{yi}(t-\tau_y)|^2\right),$$

$$2\delta_{vi}^T(t)p_{i1}(t) \le q_2\left(|\delta_{vi}(t)|^2 + |\delta_{vi}(t-\tau_v)|^2\right),$$

$$2\delta_{vi}^T(t)p_{i2}(t) \le q_1\left(|\delta_{vi}(t)|^2 + |\delta_{yi}(t-\tau_y)|^2\right).$$

It can be obtained that $\sum_{i=1}^N \delta_{yi}^T(t)p_{03}(t) = 0$, $\sum_{i=1}^N \delta_{vi}^T(t)p_{03}(t) = 0$, $|\delta_y(t)|^2 = |\overline{\delta}_y(t)|^2$, and $|\delta_v(t)|^2 = |\overline{\delta}_v(t)|^2$. Hence,

$$\begin{aligned}
2\overline{\delta}_y^T(t)(\phi^T \otimes I_n)\bar{F}(t) &= 2\delta_y^T(t)(\phi\phi^T \otimes I_n)[(I_N - J_N)\otimes I_n]F(t)\\
&= 2\delta_y^T(t)\bar{F}(t) = 2\sum_{i=1}^N \left(\delta_{yi}^T(t)p_{i1}(t) + \delta_{yi}^T(t)p_{i2}(t)\right)\\
&\le (q_1+q_2)|\overline{\delta}_y(t)|^2 + q_1|\overline{\delta}_y(t-\tau_y)|^2 + q_2|\overline{\delta}_v(t-\tau_v)|^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
2\overline{\delta}_v^T(t)(\phi^T \otimes I_n)\bar{F}(t) &= 2\sum_{i=1}^N \left(\delta_{vi}^T(t)p_{i1}(t) + \delta_{vi}^T(t)p_{i2}(t)\right)\\
&\le (q_1+q_2)|\overline{\delta}_v(t)|^2 + q_1|\overline{\delta}_y(t-\tau_y)|^2 + q_2|\overline{\delta}_v(t-\tau_v)|^2.
\end{aligned}$$

The two inequalities aforementioned produce

$$\begin{aligned}
J_2(t) &\le (q_1+q_2)\left(\theta|\overline{\delta}_y(t)|^2 + |\overline{\delta}_v(t)|^2\right)\\
&\quad + (\theta+1)\left[q_1|\overline{\delta}_y(t-\tau_y)|^2 + q_2|\overline{\delta}_v(t-\tau_v)|^2\right].
\end{aligned}\tag{13}$$

By the inequality $2x^Ty \le \frac{1}{\varepsilon}|x|^2 + \varepsilon|y|^2$, $\varepsilon > 0$, $x, y \in \mathbb{R}^m$, we have

$$J_3(t) \le \tau_1\overline{\delta}^T(t)(L^TPL\otimes I_n)\overline{\delta}(t) + \int_{t-\tau_1}^t \overline{\delta}^T(s)\left(L_1^TPL_1 \otimes I_n\right)\overline{\delta}(s)\mathrm{d}s \tag{14}$$

and

$$J_4(t) \le \tau_2\overline{\delta}^T(t)(L^TPL\otimes I_n)\overline{\delta}(t) + \int_{t-\tau_2}^t \overline{\delta}^T(s)\left(L_2^TPL_2 \otimes I_n\right)\overline{\delta}(s)\mathrm{d}s. \tag{15}$$

It can be seen that

$$J_5(t) = -2k_1\int_{t-\tau_1}^t \overline{\delta}_y^T(s)\mathrm{d}s(\Lambda\phi^T \otimes I_n)\bar{F}(t),$$

$$J_6(t) = -2k_2\int_{t-\tau_2}^t \overline{\delta}_v^T(s)\mathrm{d}s(\Lambda\phi^T \otimes I_n)\bar{F}(t).$$

Note that $\overline{\delta}_y = \phi^T\delta_y$, $\Lambda = \phi^T\mathcal{L}\phi$, $\phi\phi^T = I_N - J_N$, $(I_N - J_N)^2 = (I_N - J_N)$, $(I_N - J_N)\mathcal{L} = \mathcal{L} = \mathcal{L}(I_N - J_N)$, and $\mathcal{L}^T\mathbf{1}_N = 0$. Then, we obtain

$$\begin{aligned}
J_5(t) &= -2k_1\left(\int_{t-\tau_1}^{t}\overline{\delta}_y(s)\mathrm{d}s\right)^T(\phi^T\mathcal{L}^T\phi\phi^T\otimes I_n)\bar{F}(t)\\
&= -2k_1\left(\int_{t-\tau_1}^{t}\delta_y(s)\mathrm{d}s\right)^T(\mathcal{L}^T\otimes I_n)\bar{F}(t)\\
&= -2k_1\left(\int_{t-\tau_1}^{t}\delta_y(s)\mathrm{d}s\right)^T(\mathcal{L}^T\otimes I_n)[p_1(t)+p_2(t)]\\
&\le 2k_1^2(q_1+q_2)\int_{t-\tau_1}^{t}\overline{\delta}_y^T(s)(\Lambda^2\otimes I_n)\overline{\delta}_y(s)|^2\mathrm{d}s + \left(\frac{1}{q_2}|p_1(t)|^2+\frac{1}{q_1}|p_2(t)|^2\right)\tau_1\\
&\le k_1^2(q_1+q_2)\int_{t-\tau_1}^{t}\overline{\delta}_y^T(s)(\Lambda^2\otimes I_n)\overline{\delta}_y(s)\mathrm{d}s + q_1\tau_1|\overline{\delta}_y(t-\tau_y)|^2+q_2\tau_1|\overline{\delta}_v(t-\tau_v)|^2,
\end{aligned}$$

and similarly,

$$J_6(t) \le k_2^2(q_1+q_2)\int_{t-\tau_2}^{t}\overline{\delta}_v^T(s)(\Lambda^2\otimes I_n)\overline{\delta}_v(s)\mathrm{d}s + q_1\tau_2|\overline{\delta}_y(t-\tau_y)|^2+q_2\tau_2|\overline{\delta}_v(t-\tau_v)|^2.$$

Then, we obtain

$$\begin{aligned}
J_5(t)+J_6(t) \le{}& k_1^2(q_1+q_2)\int_{t-\tau_1}^{t}\overline{\delta}_y^T(s)(\Lambda^2\otimes I_n)\overline{\delta}_y(s)\mathrm{d}s + q_1(\tau_1+\tau_2)|\overline{\delta}_y(t-\tau_y)|^2\\
&+k_2^2(q_1+q_2)\int_{t-\tau_2}^{t}\overline{\delta}_v^T(s)(\Lambda^2\otimes I_n)\overline{\delta}_v(s)\mathrm{d}s + q_2(\tau_1+\tau_2)|\overline{\delta}_v(t-\tau_v)|^2.
\end{aligned}\tag{16}$$

Combining (12) to (16) yields

$$\begin{aligned}
\mathfrak{L}V_1(\overline{\delta}_t) \le{}& S_1(\overline{\delta}(t)) + \int_{t-\tau_1}^{t}\overline{\delta}^T(s)\left(L_1^TPL_1\otimes I_n\right)\overline{\delta}(s)\mathrm{d}s + \int_{t-\tau_2}^{t}\overline{\delta}^T(s)\left(L_2^TPL_2\otimes I_n\right)\overline{\delta}(s)\mathrm{d}s\\
&+(\theta+1)\left[q_1|\overline{\delta}_y(t-\tau_y)|^2+q_2|\overline{\delta}_v(t-\tau_v)|^2\right] + k_1^2(q_1+q_2)\int_{t-\tau_1}^{t}\overline{\delta}_y^T(s)(\Lambda^2\otimes I_n)\overline{\delta}_y(s)\mathrm{d}s\\
&+k_2^2(q_1+q_2)\int_{t-\tau_2}^{t}\overline{\delta}_v^T(s)(\Lambda^2\otimes I_n)\overline{\delta}_v(s)\mathrm{d}s + 2\frac{N-1}{N}k_1^2\sigma_1^2\overline{\delta}_y^T(t-\tau_1)[\Lambda\otimes I_n]\overline{\delta}_y(t-\tau_1)\\
&+2\frac{N-1}{N}k_2^2\sigma_2^2\overline{\delta}_v^T(t-\tau_2)[\Lambda\otimes I_n]\overline{\delta}_v(t-\tau_2) + (\tau_1+\tau_2)\left(q_1|\overline{\delta}_y(t-\tau_y)|^2+q_2|\overline{\delta}_v(t-\tau_v)|^2\right),
\end{aligned}\tag{17}$$

where

$$S_1(\overline{\delta}(t)) = \overline{\delta}^T(t)\left(\left(L^TP+PL+L^TPL(\tau_1+\tau_2)\right)\otimes I_n\right)\overline{\delta}(t) + (q_1+q_2)\left(\theta|\overline{\delta}_y(t)|^2+|\overline{\delta}_v(t)|^2\right).$$

Hence, from the definition of $V(\cdot)$, we obtain that

$$\mathfrak{L}V(\overline{\delta}_t) \le \overline{\delta}^T(t)(Q\otimes I_n)\overline{\delta}(t),\tag{18}$$

where $Q = L^TP+PL+L^TPL(\tau_1+\tau_2)+L_1^TPL_1\tau_1+L_2^TPL_2\tau_2+D$ with $D = \begin{bmatrix} d_{11}(\theta) & 0\\ 0 & d_{22}(\theta)\end{bmatrix}$, $d_{11}(\theta) = (2q_1+q_2)\theta I_{N-1}+(\tau_1+\tau_2+1)q_1I_{N-1}+(q_1+q_2)k_1^2\Lambda^2\tau_1+2\frac{N-1}{N}k_1^2\sigma_1^2\Lambda$, and $d_{22}(\theta) = (q_2\theta+2q_2+q_1+q_2(\tau_1+\tau_2))I_{N-1}+(q_1+q_2)k_2^2\Lambda^2\tau_2+2\frac{N-1}{N}k_2^2\sigma_2^2\Lambda$. In the following, we show that conditions (3) and (4) can guarantee $P > 0$ and $Q < 0$.

Step 4: Stability and consensus analysis under degenerate Lyapunov functional. It can be obtained that

$$L^TP+PL = \begin{bmatrix} -2k_1\theta\Lambda & (\mu-k_1-\theta k_2)\Lambda\\ (\mu-k_1-\theta k_2)\Lambda & 2(\theta I_{N-1}-k_2\Lambda)\end{bmatrix},$$

$$L^TPL = \begin{bmatrix} k_1^2\Lambda^2 & k_1k_2\Lambda^2-k_1\theta\Lambda\\ k_1k_2\Lambda^2-k_1\theta\Lambda & (\mu-k_2\theta)\Lambda-k_2\theta\Lambda+k_2^2\Lambda^2\end{bmatrix},$$

$$L_1^TPL_1 = \begin{bmatrix} k_1^2\Lambda^2 & 0\\ 0 & 0\end{bmatrix} \text{ and } L_2^TPL_2 = \begin{bmatrix} 0 & 0\\ 0 & k_2^2\Lambda^2\end{bmatrix}.$$

Let $\mu = k_1 + k_2\theta + k_1\theta(\tau_1 + \tau_2)$. Note that

$$\begin{bmatrix} 0 & k_1k_2\Lambda^2 \\ k_1k_2\Lambda^2 & 0 \end{bmatrix} \le \begin{bmatrix} k_1^2\Lambda^2 & 0 \\ 0 & k_2^2\Lambda^2 \end{bmatrix}.$$

Therefore, we have

$$Q \le \begin{bmatrix} s_{11}(\theta) & 0 \\ 0 & s_{22}(\theta) \end{bmatrix},$$

where $s_{11}(\theta) = d_{11}(\theta) - 2k_1\theta\Lambda + 2k_1^2\Lambda^2(\tau_1 + \tau_2) + k_1^2\Lambda^2\tau_1$ and $s_{22}(\theta) = d_{22}(\theta) + 2(\theta I_{N-1} - k_2\Lambda) + (\mu - 2k_2\theta)\Lambda(\tau_1 + \tau_2) + k_2^2\Lambda^2(\tau_1 + 2\tau_2)$. Then, we need $s_{11}(\theta) < 0$ and $s_{22}(\theta) < 0$. It is easy to verify that $s_{11}(\theta) < 0$ for $\theta > \theta_1 := \frac{\lambda_2}{2k_1\lambda_2 - 2q_1 - q_2}[((3 + q_1 + q_2)\tau_1 + 2\tau_2)k_1^2\lambda_N + (\tau_1 + \tau_2 + 1)q_1\lambda_2^{-1} + 2k_1^2\frac{N-1}{N}\bar{\sigma}_1^2]$ and $s_{22}(\theta) < 0$ for $\theta < \theta_2 := \frac{2\lambda_2k_2(1-k_2\frac{N-1}{N}\bar{\sigma}_2^2)-\lambda_2k_1(\tau_1+\tau_2)-\lambda_2\lambda_Nk_2^2[(q_1+q_2+2)\tau_2+\tau_1]-q_1-q_2(\tau_1+\tau_2+2)}{2+q_2+k_1\lambda_N(\tau_1+\tau_2)^2}$. Note that conditions (3) and (4) guarantee $\theta_1 < \theta_2$, and then $s_{11}(\theta) < 0$ and $s_{22}(\theta) < 0$ for $\theta \in (\theta_1, \theta_2)$. That is, $Q < 0$ for $\theta \in (\theta_1, \theta_2)$. We now show that the choice $\mu = k_1 + k_2\theta + k_1\theta(\tau_1 + \tau_2)$ with $\theta \in (\theta_1, \theta_2)$ can still guarantee the matrix P to be positive definite. From $\theta^2 < \mu\lambda_2$ and $\mu = k_1 + \theta k_2 + k_1\theta(\tau_1 + \tau_2)$, it is enough to show that for $\theta \in (\theta_1, \theta_2)$, $\theta^2 - \theta\lambda_2(k_2 + k_1(\tau_1 + \tau_2)) - k_1\lambda_2 < 0$. This can be guaranteed under the condition $0 < \theta < \theta^*$, where

$$\theta^* = \left[\lambda_2\left(k_2 + k_1(\tau_1 + \tau_2)\right) + \sqrt{\lambda_2^2(k_2 + k_1(\tau_1 + \tau_2))^2 + 4k_1\lambda_2}\right]/2. \tag{19}$$

It is easy to see that $\theta_2 < \theta^*$. That is, for any $\theta \in (\theta_1, \theta_2)$, we have $P > 0$ and $Q < 0$. It is easy to compute form Hypothesis 3 that $|\widetilde{F}(t)| \le C_1(|\overline{\delta}(t-\tau_y)| + |\overline{\delta}(t-\tau_v)|)$. This, together with Hypothesis 2, implies that all the coefficients satisfy the linear growth condition. Therefore, by stochastic stability theorems (theorems 4.3 and 4.4 in the work of Zong et al[42]), we know that

$$e^{\gamma t}\mathbb{E}|\overline{\delta}(t)|^2 < C \text{ and } \limsup_{t\to\infty}\frac{1}{t}\log|\overline{\delta}(t)| < -\frac{\gamma}{2}, \quad \text{a.s.}$$

for certain $C, \gamma > 0$. Note that $\delta_{yi}(t) = y_i - \frac{1}{N}\sum_{j=1}^N y_j(t)$ and $\delta_{vi}(t) = v_i - \frac{1}{N}\sum_{j=1}^N v_j(t)$, $i = 1, \ldots, N$, and then for $i \neq j$, $|y_j(t) - y_i(t)| \le |\delta_{yj}(t)| + |\delta_{yi}(t)|$ and $|v_j(t) - v_i(t)| \le |\delta_{vj}(t)| + |\delta_{vi}(t)|$. Therefore, we can obtain mean square and almost sure consensus. □

Remark 3. For pure delayed second-order stochastic systems such as (5), little is known about the stochastic stability conditions. In the proof of Theorem 1, we introduce the degenerate Lyapunov functional $V(\cdot)$ to get the stability analysis. Here, the degenerate Lyapunov functional[42] denotes a classes of functionals without satisfying $V(\varphi) \ge c|\varphi(0)|^p$ for all continuous functions φ defined on $[-\tau, 0]$.

Remark 4. Theorem 1 implies that, for any given measurement delays τ_1, τ_2 and noise intensities $\bar{\sigma}_1, \bar{\sigma}_2$, if the growth rates of the intrinsic dynamics q_1 and q_2 are small enough, then there are k_1 and k_2 such that mean square and almost sure consensus can be obtained. In fact, for any given τ_1, τ_2 and noise intensities $\bar{\sigma}_1, \bar{\sigma}_2$, we can first fix k_2 such that $2\lambda_2k_2(1 - k_2\frac{N-1}{N}\bar{\sigma}_2^2) > 0$; then, if q_1 and q_2 are small enough, then we can choose k_1 small such that $2k_1\lambda_2 > 2q_1 + q_2$ and $\rho(k_1) < \varrho(k_2)$.

If the intrinsic dynamics vanish, ie, $q_1 = q_2 = 0$, then Theorem 1 gives the following corollary.

Corollary 1. *Suppose that Hypotheses 1, 2, and 3 hold and* $q_1 = q_2 = 0$. *If* $\rho(k_1) < \varrho(k_2)$, *where* $\rho(k_1) = \frac{(2+k_1\lambda_N(\tau_1+\tau_2)^2)}{2}[(3\tau_1 + 2\tau_2)\lambda_N + 2\frac{N-1}{N}\bar{\sigma}_1^2]k_1 + \lambda_2k_1(\tau_1 + \tau_2)$, $\varrho(k_2) = 2\lambda_2k_2(1 - k_2\frac{N-1}{N}\bar{\sigma}_2^2 - 0.5(2\tau_2 + \tau_1)k_2\lambda_N)$, *then the protocol (2) solves mean square and almost sure consensus.*

Remark 5. For the case without intrinsic dynamics ($q_1 = q_2 = 0$), the previous work of Zong et al[30] studied stochastic consensus problems under the condition $\tau_1 = \tau_2$, ie, the delays in position measurements and velocity measurements are equal, while Corollary 1 relaxes this condition by considering different measurement delays. Moreover, Corollary 1 shows that, for any noise intensity and measurement delay, mean square and almost sure consensus can be achieved by choosing the appropriate control gains k_1 and k_2. In fact, for any given τ_1, τ_2, and σ_{lji}, we first choose k_2 satisfying $k_2 < k_2^* := (\frac{N-1}{N}\bar{\sigma}_2^2 + 0.5(2\tau_2 + \tau_1)\lambda_N)^{-1}$, which can assure $\varrho(k_2) > 0$ for $k_2 < k_2^*$. Then, fix k_2 and choose a sufficiently small k_1 that $\rho(k_1) < \varrho(k_2)$ since $\lim_{k_1\to 0}\rho(k_1) = 0$. Therefore, mean square and almost sure consensus can be obtained.

If all the noises and delays in the measurements vanish, then we obtain the following corollary, which implies that, for arbitrary Lipschitz constants q_1 and q_2, multiagent consensus can be solved by choosing sufficiently large control gains.

Corollary 2. *Suppose that Hypotheses 1, 2, and 3 hold and $\tau_1 = \tau_2 = \bar{\sigma}_1 = \bar{\sigma}_2 = 0$. If the control gains k_1 and k_2 satisfy*

$$k_1 > \frac{2q_1 + q_2}{2\lambda_2}, k_2 > \frac{q_1 + 2q_2}{2\lambda_2},$$

and

$$(2 + q_2)q_1 < (2\lambda_2 k_2 - q_1 - 2q_2)(2\lambda_2 k_1 - 2q_1 - q_2),$$

then the protocol (2) solves the deterministic consensus.

Remark 6. Corollary 6 improves theorem 1 and corollary 2 in the work of Yu et al[34] in the case of undirected graphs. For example, we assume $N = 4, q_1 = 0.2, q_2 = 0.1$ and take $k_1 = 1.2, k_2 = 1.1$, then $\lambda_2 = 0.5858$. Then we can see that $0.5858 = \lambda_2 < 0.5 * (q_1/k_1 + k_1/k_2^2 + q_1/k_2 + \sqrt{(q_1/k_1 - k_1/k_2^2 - q_1/k_2)^2 + (k_1 + k_2)^2 * q_2^2/(k_1^2 * k_2^2)}) = 1.1810$ and $\lambda_2 < \frac{k_1}{k_2^2}$. That is, the sufficient conditions in theorem 1 and corollary 2 in the work of Yu et al[34] are defied and whether the consensus can be solved is unknown from the work of the aforementioned authors.[34] However, we can see that $0.42 = (2 + q_2)q_1 < (2\lambda_2 k_2 - q_1 - 2q_2)(2\lambda_2 k_1 - 2q_1 - q_2) = 0.8051$. That is, conditions in Corollary 6 hold and the consensus is solved.

For the case with multiplicative noises alone[31] or intrinsic dynamics alone (Corollary 6), mean square and almost sure consensus can be definitely solved by protocol (2) with appropriate k_1 and k_2. However, the coexistence of multiplicative noises and intrinsic dynamics may weaken or destroy the consentability with respect to linear protocols. From Theorem 1, $\rho(k_1) < \varrho(k_2)$ implies $\varrho(k_2) > 0$, which means $k_2 < \frac{2}{2\frac{N-1}{N}\bar{\sigma}_2^2 + [(q_1+q_2+2)\tau_2 + \tau_1]\lambda_N}$. At the same time, $\rho(k_1) < \varrho(k_2)$ implies $k_2 > \frac{q_1+q_2(\tau_1+\tau_2+2)}{2\lambda_2}$ at least. Hence, the condition $\rho(k_1) < \varrho(k_2)$ cannot hold for large $\bar{\sigma}_1, \bar{\sigma}_2, q_1$, and q_2. That is, the consensus may not be solved by $u(t)$ for any k_1, k_2 when multiplicative noises and intrinsic dynamics exist simultaneously with large intensities and large growth rates, respectively. This is revealed clearly in Theorem 2 and Remark 3 in Section 2.2.

2.2 | Second-order consensus under linear intrinsic dynamics

From the proof of Theorem 1, we can see that the intrinsic dynamics and the delays in intrinsic dynamics and measurements increase the difficulty and complexity of consensus analysis. In the aforementioned statement, we only get sufficient conditions. Particularly, if all the delays vanish and the intrinsic dynamics are linear, we can find more simpler sufficient conditions for mean square and almost sure consensus, and necessary conditions for mean square consensus. We need the following hypothesis.

Hypothesis 4. *For each (j, i), $g_{1ji}(0) = g_{2ji}(0) = 0$ and there exist positive constants σ_1, σ_2, $\underline{\sigma}_1$, and $\underline{\sigma}_2$ such that for all $y_1, y_2 \in \mathbb{R}$,*

$$\underline{\sigma}_1|y_1 - y_2| \le \left|g_{1ji}(y_1) - g_{1ji}(y_2)\right| \le \sigma_1|y_1 - y_2|$$

and

$$\underline{\sigma}_2|y_1 - y_2| \le \left|g_{2ji}(y_1) - g_{2ji}(y_2)\right| \le \sigma_2|y_1 - y_2|.$$

Theorem 2. *Suppose Hypotheses 1, 2, and 4 hold, $\tau_y = \tau_v = \tau_1 = \tau_2 = 0$, and $f(y, v, t) = q_1 y + q_2 v$ with $q_1 \ge 0$ and $q_2 \ge 0$. If there exist k_1 and k_2 such that*

$$k_1 > \frac{q_1}{\lambda_2}, k_2 > \frac{q_2}{\lambda_2}, \tag{20}$$

and

$$\frac{k_1^2\sigma_1^2\frac{N-1}{N}}{k_1 - \frac{q_1}{\lambda_2}} < k_2\lambda_2 - q_2 - \frac{N-1}{N}k_2^2\sigma_2^2\lambda_2, \tag{21}$$

then the protocol (2) solves mean square and almost sure consensus. Moreover, if the protocol $u(t)$ solves mean square consensus, then k_1 and k_2 must satisfy (20) and

$$k_2\lambda_N - \frac{N-1}{N}k_2^2\underline{\sigma}_2^2\lambda_N > q_2. \tag{22}$$

Proof. For linear intrinsic dynamics, (6) can be simplified as the following stochastic differential equation (SDE):

$$d\bar{\delta}(t) = (L_3 \otimes I_n)\bar{\delta}(t)dt + d\bar{M}_3(t) + d\bar{M}_4(t), \tag{23}$$

where $L_3 = \begin{bmatrix} 0 & I_{N-1} \\ q_1 I_{N-1} - k_1\Lambda & q_2 I_{N-1} - k_2\Lambda \end{bmatrix}$, $\bar{M}_3(t)$, and $\bar{M}_4(t)$ are defined in (6) with $\tau_1 = \tau_2 = 0$. Then, the consensus problem is converted into the stability problem of SDE (23). We choose

$$P = \begin{bmatrix} k_1\Lambda - q_1 I_{N-1} + \theta(k_2\Lambda - q_2 I_{N-1}) & \theta I_{N-1} \\ \theta I_{N-1} & I_{N-1} \end{bmatrix}. \tag{24}$$

Here, $P > 0$ for $\theta \in [0, \theta^*)$, where

$$\theta^* = \frac{k_2\lambda_2 - q_2 + \sqrt{(k_2\lambda_2 - q_2)^2 + 4(k_1\lambda_2 - q_1)}}{2}.$$

Choose the following Lyapunov function:

$$V_3(\bar{\delta}(t)) = \bar{\delta}^T(t)(P \otimes I_n)\bar{\delta}(t). \tag{25}$$

Then, one can obtain that

$$\mathfrak{L}V_3(\bar{\delta}(t)) \leq \bar{\delta}^T(t)(S \otimes I_n)\bar{\delta}(t),$$

where $S = 2\begin{bmatrix} s_{11}(\theta) & 0 \\ 0 & s_{22}(\theta) \end{bmatrix}$ with $s_{11}(\theta) = -(k_1\Lambda - q_1 I_{N-1})\theta + \frac{N-1}{N}k_1^2\sigma_1^2\Lambda$ and $s_{22}(\theta) = \theta I_{N-1} - k_2\Lambda + q_2 I_{N-1} - \frac{N-1}{N}k_2^2\sigma_2^2\Lambda$. Hence, $s_{11}(\theta) < 0$ for $\theta > \theta_1 := \frac{k_1^2\sigma_1^2\frac{N-1}{N}}{k_1 - \frac{q_1}{\lambda_2}}$ and $s_{22}(\theta) < 0$ for $\theta < \theta_2 := k_2\lambda_2 - q_2 - \frac{N-1}{N}k_2^2\sigma_2^2\lambda_2$. It is easy to see that $\theta_2 < \theta^*$ and condition (21) implies $\theta_1 < \theta_2$. Hence, the choice $\theta \in (\theta_1, \theta_2)$ guarantees that $P > 0$ and $S < 0$. Therefore, by stochastic stability theorems (theorems 4.2 and 4.4 in the work of Mao[43]), we know that mean square and almost sure consensus follow.

Note that mean square consensus is equivalent to mean square stability of (23). Note that Brownian motions do not contribute positively to the mean square stability of the closed-loop system, ie, the unstable system cannot be mean square stabilized by Brownian motions (see the work of Zong et al[44]). Hence, in order for the mean square stability of SDE (23), the deterministic part must be stable. That is, the matrix L_3 must be Hurwitz, which implies condition (20). We choose the Lyapunov function as follows:

$$V_4(t) = \bar{\delta}_y^T(t)\left[(k_1\Lambda - q_1 I_{N-1}) \otimes I_n\right]\bar{\delta}_y(t) + |\bar{\delta}_v(t)|^2. \tag{26}$$

Applying the Itô formula to $V_4(t)$, we have

$$\begin{aligned} dV_4(t) = &-2\bar{\delta}_v^T(t)\left[(k_2\Lambda + q_2 I_{N-1}) \otimes I_n\right]\bar{\delta}_v(t)dt + d\langle\bar{M}_1\rangle(t) \\ &+ d\langle\bar{M}_2\rangle(t) + 2\bar{\delta}_v^T(t)d\left[\bar{M}_1(t) + \bar{M}_2(t)\right], \end{aligned} \tag{27}$$

where $\bar{M}_1(t)$ and $\bar{M}_2(t)$ are defined in (5), $\langle\bar{M}_1\rangle(t) = k_1^2\sum_{i,j=1}^N a_{ij}\int_0^t |b_{1ij}(s)|^2 ds$, and $\langle\bar{M}_2\rangle(t) = k_2^2\sum_{i,j=1}^N a_{ij}\int_0^t |b_{2ij}(s)|^2 ds$. Similar to (12), we can get

$$d\langle\bar{M}_1\rangle(t) \geq 2k_1^2\underline{\sigma}_2^2\frac{N-1}{N}\bar{\delta}_y^T(t)(\Lambda \otimes I_n)\bar{\delta}_y(t)dt \tag{28}$$

and

$$d\langle\bar{M}_2\rangle(t) \geq 2k_2^2\underline{\sigma}_2^2\frac{N-1}{N}\bar{\delta}_v^T(t)(\Lambda \otimes I_n)\bar{\delta}_v(t)dt. \tag{29}$$

Hence, taking expectations on the both sides of (27), we obtain

$$\mathbb{E}V_4(t) \geq V_4(0) + 2\mathbb{E}\int_0^t \bar{\delta}_v^T(s)\left[h(k_2, \Lambda) \otimes I_n\right]\bar{\delta}_v(s)ds, \tag{30}$$

where $h(k_2, \Lambda) = q_2 I_{N-1} - k_2\Lambda + k_2^2\underline{\sigma}^2\frac{N-1}{N}\Lambda$, since $\langle\bar{M}_1\rangle(t) \geq 0$. Let $h_i(k_2) = q_2 I_{N-1} - k_2\lambda_i + k_2^2\underline{\sigma}^2\frac{N-1}{N}\lambda_i$, $i = 2, \dots, N$. Now, if condition (22) is defied, then $h_i(k_2) \geq 0$ for any $i = 2, \dots, N$. Hence, $h(k_2, \Lambda) \geq 0$, and then we must have $\liminf_{t\to 0}\mathbb{E}V_4(t) \geq V_4(0) > 0$ for $\bar{\delta}(0) \neq 0$. This is in conflict with the definition of the mean square consensus. Therefore, the necessity of condition (22) follows. □

Remark 7. Theorem 2 gives the explicit consensus conditions of mean square and almost sure consensus. At the same time, the necessary condition (22) implies that the gains k_1 and k_2 for mean square consensus may not exist. In fact, if $q_2, \underline{\sigma}_2$ are large and satisfies $4q_2\frac{N-1}{N}\underline{\sigma}_2^2 > \lambda_N$, then $k_2\lambda_N - q_2 - \frac{N-1}{N}k_2^2\sigma_2^2\lambda_N \leq 0$, for any $k_2 \in \mathbb{R}$. That is, (22) is violated, and then mean square consensus cannot be achieved for any $k_1, k_2 \in \mathbb{R}$. The necessity of (22) is also revealed in the simulation section (Section 4), where an example is given to see that mean square consensus cannot be achieved without (22).

In Theorem 2, we study the linear case with $q_1 \geq 0$ and $q_2 \geq 0$. If $q_1 < 0$ and $q_2 < 0$, then each single original second-order system is stable, and then the noisy input control with small control gains does not affect such stability.[30] In the following, we will consider the cases $q_1 < 0, q_2 \geq 0$ and $q_1 \geq 0, q_2 < 0$.

Theorem 3. *Suppose Hypotheses 1, 2, and 4 hold,* $\tau_y = \tau_v = \tau_1 = \tau_2 = 0$, *and* $f(y, v, t) = q_1 y + q_2 v$ *with* $q_1 < 0$ *and* $q_2 \geq 0$. *If* k_2 *satisfies*

$$0 < k_2\lambda_2 - q_2 - \frac{N-1}{N}k_2^2\sigma_2^2\lambda_2, \tag{31}$$

then the protocol (2) with $k_1 = 0$ *solves mean square and almost sure consensus. Moreover, if the protocol* $u(t)$ *with* $k_1 = 0$ *solves mean square consensus, then* k_2 *must satisfy (22) and*

$$k_2 > \frac{q_2}{\lambda_2}. \tag{32}$$

Proof. Under the protocol $u(t)$ with $k_1 = 0$, (6) can be simplified as

$$d\bar{\delta}(t) = (L_3 \otimes I_n)\bar{\delta}(t)dt + d\bar{M}_4(t), \tag{33}$$

where $L_3 = \begin{bmatrix} 0 & I_{N-1} \\ q_1 I_{N-1} & q_2 I_{N-1} - k_2\Lambda \end{bmatrix}$. Then, the consensus problem is converted into the stability problem of SDE(33). We choose P defined by (24) with $k_1 = 0$. Note that (31) implies (32). Hence, $P > 0$ for $\theta \in [0, \theta^*)$, where $\theta^* = \frac{k_2\lambda_2 - q_2 + \sqrt{(k_2\lambda_2 - q_2)^2 - 4q_1}}{2}$. Considering the Lyapunov function (25), one can obtain that

$$\mathfrak{L}V_3(\bar{\delta}(t)) \leq \bar{\delta}^T(t)(S \otimes I_n)\bar{\delta}(t),$$

where $S = 2\begin{bmatrix} s_{11}(\theta) & 0 \\ 0 & s_{22}(\theta) \end{bmatrix}$ with $s_{11}(\theta) = -q_1\theta I_{N-1}$ and $s_{22}(\theta) = \theta I_{N-1} - k_2\Lambda + q_2 I_{N-1} - \frac{N-1}{N}k_2^2\sigma_2^2\Lambda$. Hence, for $s_{11}(\theta) < 0$ for $\theta > 0$ and $s_{22}(\theta) < 0$ for $\theta < \theta_2 := k_2\lambda_2 - q_2 - \frac{N-1}{N}k_2^2\sigma_2^2\lambda_2$, it is easy to see that $\theta_2 < \theta^*$ and condition (31) implies $0 < \theta_2$. Hence, the choice $\theta \in (0, \theta_2)$ guarantees that $P > 0$ and $S < 0$. Therefore, by stochastic stability theorems (theorems 4.2 and 4.4 in the work of Mao[43]), we know that mean square and almost sure consensus follow.

Note that mean square consensus is equivalent to mean square stability of (33). Hence, in order for the mean square stability of SDE (33), the matrix L_3 must be Hurwitz, which implies condition (32). Considering the Lyapunov function $V_4(t)$ defined by (26) with $k_1 = 0$ and applying the Itô formula to $V_4(t)$, we have

$$dV_4(t) = -2\bar{\delta}_v^T(t)\left[(k_2\Lambda + q_2 I_{N-1}) \otimes I_n\right]\bar{\delta}_v(t)dt + d\langle\bar{M}_2\rangle(t) + 2\bar{\delta}_v^T(t)d\bar{M}_2(t), \tag{34}$$

where $\bar{M}_2(t)$ is defined in (5) and $\langle\bar{M}_2\rangle(t) = k_2^2\sum_{i,j=1}^N a_{ij}\int_0^t |b_{2ij}(s)|^2 ds$ satisfies (29). Hence, taking expectations on the both sides of (34), we obtain (30). Then, by the similar skills in proving Theorem 2, we can obtain the necessity of (22). □

Theorem 4. *Suppose Hypotheses 1, 2, and 4 hold,* $\tau_y = \tau_v = \tau_1 = \tau_2 = 0$, *and* $f(y, v, t) = q_1 y + q_2 v$ *with* $q_1 \geq 0$ *and* $q_2 < 0$. *If* k_1 *satisfies*

$$k_1 > \frac{q_1}{\lambda_2} \tag{35}$$

and

$$\frac{k_1^2\sigma_1^2\frac{N-1}{N}}{k_1 - \frac{q_1}{\lambda_2}} < -q_2, \tag{36}$$

then the protocol (2) with $k_2 = 0$ solves mean square and almost sure consensus. Moreover, if the protocol $u(t)$ with $k_2 = 0$ solves mean square consensus, then k_1 must satisfy (35) and

$$\frac{k_1^2\sigma_1^2\frac{N-1}{N}}{k_1-\frac{q_1}{\lambda_N}} < -q_2. \tag{37}$$

Proof. Under the protocol (2) with $k_2 = 0$, (6) is be simplified as

$$d\bar{\delta}(t) = (L_3 \otimes I_n)\bar{\delta}(t)dt + d\bar{M}_3(t), \tag{38}$$

where $L_3 = \begin{bmatrix} 0 & 1 \\ q_1 I_{N-1} - k_1\Lambda & q_2 I_{N-1} \end{bmatrix}$ and $\bar{M}_3(t)$ is defined in (6) with $\tau_1 = 0$. Then, the consensus problem is converted into the stability problem of SDE (38). We choose

$$P = \begin{bmatrix} k_1\Lambda - q_1 I_{N-1} - \theta q_2 I_{N-1} & \theta I_{N-1} \\ \theta I_{N-1} & I_{N-1} \end{bmatrix}.$$

Here, $P > 0$ for $\theta \in [0, \theta^*)$, where $\theta^* = \frac{-q_2+\sqrt{q_2^2+4(k_1\lambda_2-q_1)}}{2}$. Choose the following Lyapunov function:

$$V_3\left(\bar{\delta}(t)\right) = \bar{\delta}^T(t)(P \otimes I_n)\bar{\delta}(t). \tag{39}$$

Then, one can obtain that

$$\mathfrak{L}V_3(\bar{\delta}(t)) \leq \bar{\delta}^T(t)(S \otimes I_n)\bar{\delta}(t),$$

where $S = 2\begin{bmatrix} s_{11}(\theta) & 0 \\ 0 & s_{22}(\theta) \end{bmatrix}$ with $s_{11}(\theta) = -(k_1\Lambda - q_1 I_{N-1})\theta + \frac{N-1}{N}k_1^2\sigma_1^2\Lambda$ and $s_{22}(\theta) = \theta I_{N-1} + q_2 I_{N-1}$. Hence, $s_{11}(\theta) < 0$ for $\theta > \theta_1 := \frac{k_1^2\sigma_1^2\frac{N-1}{N}}{k_1-\frac{q_1}{\lambda_2}}$ and $s_{22}(\theta) < 0$ for $\theta < \theta_2 := -q_2$. It is easy to see that $\theta_2 < \theta^*$ and condition (36) implies $\theta_1 < \theta_2$. Hence, the choice $\theta \in (\theta_1, \theta_2)$ guarantees that $P > 0$ and $S < 0$. Therefore, by stochastic stability theorems (theorems 4.2 and 4.4 in the work of Mao[43]), we know that mean square and almost sure consensus follow.

The necessity of condition (35) for mean square consensus can be obtained similarly to that of (20) for mean square consensus. The remaining is to prove that mean square consensus implies (37). For the Lyapunov function (39), we can obtain

$$\mathfrak{L}V_3(\bar{\delta}(t)) \geq \bar{\delta}^T(t)(\underline{S} \otimes I_n)\bar{\delta}(t), \tag{40}$$

where $\underline{S} = 2\begin{bmatrix} \underline{s}_{11}(\theta) & 0 \\ 0 & \underline{s}_{22}(\theta) \end{bmatrix}$ with $\underline{s}_{11}(\theta) = -(k_1\Lambda - q_1 I_{N-1})\theta + \frac{N-1}{N}k_1^2\underline{\sigma}_1^2\Lambda$ and $\underline{s}_{22}(\theta) = (\theta + q_2)I_{N-1}$. If condition (37) is defied, then

$$-q_2 \leq \theta_3 := \frac{k_1^2\underline{\sigma}_1^2\frac{N-1}{N}}{k_1-\frac{q_1}{\lambda_N}} \wedge \theta^*. \tag{41}$$

It can be proved that $\underline{s}_{11}(\theta) \geq 0$ and $\underline{s}_{22}(\theta) \geq 0$ for $\theta \in [-q_2, \theta_3]$. This, together with Itô formula and (40), yields

$$\mathbb{E}V_3(\bar{\delta}(t)) \geq V_3(\bar{\delta}(0)) + \mathbb{E}\int_0^t \bar{\delta}^T(s)(\underline{S} \otimes I_n)\bar{\delta}(s)\mathrm{d}s \geq V_3(\bar{\delta}(0)).$$

Hence, we must have $\liminf_{t\to 0} \mathbb{E}V_3(\bar{\delta}(t)) \geq V_3(\bar{\delta}(0)) > 0$ for $\bar{\delta}(0) \neq 0$. This is in conflict with the definition of mean square consensus. Therefore, the necessity of condition (37) follows. □

Remark 8. Theorems 3 and 4 tell us that if the original linear second-order system is partially stable ($q_1 < 0$ or $q_2 < 0$); then, the protocol with partial relative measurements (the velocity measurement or the position measurement) is enough to solve mean square and almost sure consensus. Moreover, we find some necessary conditions for mean square consensus. Especially, for the case $q_2 < 0$, the condition (37) shows the necessary relationship between the control gain k_1 for the position measurement and the velocity coefficient q_2 in the original system, and then it is different from the necessary condition (22) in Theorems 2 and 3, which focus on the necessary relationship between the control gain k_2 for the velocity measurement and the velocity coefficient q_2 in the original system.

3 | LEADER-FOLLOWING MULTIAGENT SYSTEMS

In this section, we consider a leader-following multiagent system consisting of $N + 1$ agents where the agent indexed by 0 acts as the leader and the other agents indexed by $1, 2, \dots, N$, respectively, act as the followers. Generally, the behavior of the leader is independent of the followers. Here, x_0 denotes the state of the leader and is assumed to have linear dynamics as

$$\dot{y}_0(t) = v_0(t),\ \dot{v}_0(t) = f\left(y_0(t - \tau_y), v_0(t - \tau_v)\right). \tag{42}$$

For the ith follower, the dynamics is described by (1) with $u_i(t)$ defined by (2). Note that this is different from Section 3, since for each agent i, its neighbor set N_i may contain the leader 0. Hypotheses 1 and 2 is also deemed to include the leader 0. Considering the information flow from the leader to the followers, we denote the topology graph by $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{V}}, \widetilde{\mathcal{A}}\}$ with $\widetilde{\mathcal{V}} = \{0, 1, 2, \dots, N\}$ and $\widetilde{\mathcal{A}} = \begin{pmatrix} 0 & 0_{N\times N} \\ a_0 & \mathcal{A} \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}$, where $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, $a_0 = [a_{10}, \dots, a_{N0}]^T$, $a_{i0} = 1$ if $0 \in N_i$; otherwise, $a_{i0} = 0$. Let $D_0 = \operatorname{diag}(a_{10}, \dots, a_{N0})$. We use $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ to represent the subgraph formed by the N followers, where $\mathcal{V} = \widetilde{\mathcal{V}} \setminus \{0\}$.

Definition 2. We say that the protocol $u(t)$ solves mean square (or almost sure) leader-following consensus if it makes the $N + 1$ agents have the property that for any initial data φ and all $i \in \mathcal{V}$, $\lim_{t\to\infty} \mathbb{E}|x_i(t) - x_0(t)|^2 = 0$ (or $\lim_{t\to\infty} |x_i(t) - x_0(t)| = 0$ a.s.).

We impose the following assumption on the graph $\widetilde{\mathcal{G}}$ and its subgraph $\mathcal{G}$.

Hypothesis 5. *Assume that the graph $\widetilde{\mathcal{G}}$ contains a spanning tree and its subgraph $\mathcal{G}$ is undirected.*

Let $\mathcal{L}_0 = \mathcal{L} + D_0$. Under Assumption 5, we know that $\mathcal{L}_0$ is symmetric, and then all the eigenvalues of the matrix $\mathcal{L}_0$ are positive,[45] denoted by $\{\lambda_{0i}\}_{i=1}^N$. Hence, there exists an unitary matrix Φ such that $\Phi^T \mathcal{L}_0 \Phi = \operatorname{diag}(\lambda_{01}, \dots, \lambda_{0N}) =: \Lambda_0$. Without loss of generality, we assume $0 < \lambda_{01} \leq \dots \leq \lambda_{0N}$.

Theorem 5. *Suppose Hypotheses 1, 2, 3, and 5 hold. If the control gains k_1 and k_2 satisfy*

$$2k_1\lambda_{01} > 2q_1 + q_2 \tag{43}$$

and

$$\rho(k_1) < \varrho(k_2), \tag{44}$$

where $\rho(k_1) = \frac{(2+q_2+k_1\lambda_{0N}(\tau_1+\tau_2)^2)\lambda_{01}}{2k_1\lambda_{01}-2q_1-q_2}[((3+q_1+q_2)\tau_1+2\tau_2)k_1^2\lambda_{0N}+(\tau_1+\tau_2+1)q_1\lambda_{01}^{-1}+2k_1^2\bar{\sigma}_1^2]+\lambda_{01}k_1(\tau_1+\tau_2)+q_1+q_2(\tau_1+\tau_2+2)$ *and* $\varrho(k_2) = 2\lambda_{01}k_2(1 - k_2\bar{\sigma}_2^2) - \lambda_{01}[(q_1 + q_2 + 2)\tau_2 + \tau_1]k_2^2\lambda_{0N}$, *then the protocol (2) solves mean square and almost sure leader-following consensus.*

Proof. Let $\delta_{yi}(t) = y_i(t) - y_0(t)$, $\delta_{vi}(t) = v_i(t) - v_0$ for $i = 1, \dots, N$. Define $y(t) = [y_1^T(t), \dots, y_N^T(t)]^T$, $v(t) = [v_1^T(t), \dots, v_N^T(t)]^T$, $\delta_y(t) = [\delta_{y1}^T(t), \dots, \delta_{yN}^T(t)]^T$, and $\delta_v(t) = [\delta_{v1}^T(t), \dots, \delta_{vN}^T(t)]^T$. Let $\bar{\delta}_y(t) = \Phi^T\delta_y(t)$, $\bar{\delta}_v(t) = \Phi^T\delta_v(t)$, $\Phi(i) = \Phi^T\eta_{N,i}$, and $L = L_0 + L_1 + L_2$ with $L_0 = \begin{bmatrix} 0 & I_N \\ 0 & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 0 & 0 \\ -k_1\Lambda_0 & 0 \end{bmatrix}$, and $L_1 = \begin{bmatrix} 0 & 0 \\ 0 & -k_2\Lambda_0 \end{bmatrix}$. Substituting the protocol (2) into the system (1) and using the definitions of $\bar{\delta}_y(t)$ and $\bar{\delta}_v(t)$, we have $d\bar{\delta}_y(t) = \bar{\delta}_v(t)dt$, and

$$d\bar{\delta}_v(t) = (\Phi^T \otimes I_n)\bar{F}(t)dt - k_1(\Lambda_0 \otimes I_n)\bar{\delta}_y(t-\tau_1)dt - k_2(\Lambda_0 \otimes I_n) \otimes I_n\bar{\delta}_v(t-\tau_2)dt + d\widehat{M}_{12}(t), \tag{45}$$

where $\bar{F}(t) = F(t) - 1_N \otimes f(y_0(t-\tau_y), v_0(t-\tau_v))$, $F(t)$ is defined in the proof of Theorem 1, and $\widehat{M}_{12}(t) = k_1\sum_{i=1}^N\sum_{j=0}^N a_{ij}\int_0^t \Phi(i)f_{1ji}(\delta_{yj}(s-\tau_1) - \delta_{yi}(s-\tau_1))\mathrm{d}w_{1ji}(s) + k_2\sum_{i=1}^N\sum_{j=0}^N a_{ij}\int_0^t \Phi(i)f_{2ji}(\delta_{vj}(s-\tau_2) - \delta_{vi}(s-\tau_2))\mathrm{d}w_{2ji}(s)$. By the definition of $\bar{\delta}(t)$, we obtain

$$d\bar{\delta}(t) = \widetilde{F}(t)dt + (L_0 \otimes I_n)\bar{\delta}(t)dt + (L_1 \otimes I_n)\bar{\delta}(t-\tau_1)dt + (L_2 \otimes I_n)\bar{\delta}(t-\tau_2)dt + d\widehat{M}_{34}(t),$$

where $\widetilde{F}(t) = [0, (\phi^T \otimes I_n)\bar{F}^T(t)]^T$, $\widehat{M}_{34}(t) = \widehat{M}_3(t) + \widehat{M}_4(t)$, $\widehat{M}_3(t) = k_1\sum_{i=1}^N\sum_{j=0}^N a_{ij}\int_0^t B_{1ij}(s-\tau_1)\mathrm{d}w_{1ji}(s)$, $\widehat{M}_4(t) = k_2\sum_{i=1}^N\sum_{j=0}^N a_{ij}\int_0^t B_{2ij}(s-\tau_2)\mathrm{d}w_{2ji}(s)$, $B_{1ij}(t) = [0, b_{1ij}^T(t)]^T$, $b_{1ij}(t) = \Phi(i)f_{1ji}(\delta_{yj}(t) - \delta_{yi}(t)))$, $B_{2ij}(t) = [0, b_{2ij}^T(t)]^T$, and $b_{2ij}(t) = \Phi(i)f_{2ji}(\delta_{vj}(t) - \delta_{vi}(t))$. Let $z(t)$ be defined by (8) and choose $P = \begin{bmatrix} \mu\Lambda_0 & \theta I_N \\ \theta I_N & I_N \end{bmatrix}$ with $\mu, \theta > 0$ to be designed.

Similarly, we choose the Lyapunov functional $V(\overline{\delta}_t) = V_1(t) + V_2(t)$, where $V_1(\overline{\delta}_t) = z(t)^T(P \otimes I_n)z(t)$ and

$$\begin{aligned}
V_2(\overline{\delta}_t) = & \int_{-\tau_1}^{0} \left[\int_{t+s}^{t} |\overline{\delta}^T(\theta) \left(L_1^T P L_1 \otimes I_n\right) \overline{\delta}(\theta)\mathrm{d}\theta \right] \mathrm{d}s \\
& + \int_{-\tau_2}^{0} \left[\int_{t+s}^{t} |\overline{\delta}^T(\theta) \left(L_2^T P L_2 \otimes I_n\right) \overline{\delta}(\theta)\mathrm{d}\theta \right] \mathrm{d}s \\
& + (\theta+1) \left[q_1 \int_{t-\tau_y}^{t} \left|\overline{\delta}_y(s)\right|^2 \mathrm{d}s + q_2 \int_{t-\tau_v}^{t} \left|\overline{\delta}_v(s)\right|^2 \mathrm{d}s \right] \\
& + k_1^2(q_1+q_2) \int_{-\tau_1}^{0} \int_{t+s}^{t} \overline{\delta}_y^T(u) \left(\Lambda_0^2 \otimes I_n\right) \overline{\delta}_y(u)\mathrm{d}u\mathrm{d}s \\
& + k_2^2(q_1+q_2) \int_{-\tau_2}^{0} \int_{t+s}^{t} \overline{\delta}_v^T(u) \left(\Lambda_0^2 \otimes I_n\right) \overline{\delta}_v(u)\mathrm{d}u\mathrm{d}s \\
& + (\tau_1+\tau_2) \left(q_1 \int_{t-\tau_y}^{t} \left|\overline{\delta}_y(s)\right|^2 \mathrm{d}s + q_2 \int_{t-\tau_v}^{t} \left|\overline{\delta}_v(s)\right|^2 \mathrm{d}s \right) \\
& + 2k_1^2\sigma_1^2 \int_{t-\tau_1}^{t} \overline{\delta}_y^T(s)[\Lambda_0 \otimes I_n]\overline{\delta}_y(s)\mathrm{d}s + 2k_2^2\sigma_2^2 \int_{t-\tau_2}^{t} \overline{\delta}_v^T(s)[\Lambda_0 \otimes I_n]\overline{\delta}_v(s)\mathrm{d}s.
\end{aligned}$$

We compute the Itô operator $\mathcal{L}V_1(\cdot)$ as follows:

$$\begin{aligned}
\mathfrak{L}V_1(\overline{\delta}_t) &= 2z(t)^T(P \otimes I_n) \left[\widetilde{F}(t) + (L \otimes I_n)\overline{\delta}(t)\right] + J_1(t) \\
&= \overline{\delta}(t)^T \left[(L^T P + PL) \otimes I_n\right] \overline{\delta}(t) + \sum_{j=1}^{4} J_j(t),
\end{aligned}$$

where $\{J_i(t)\}_{i=2}^{6}$ have the same forms as these in the proof of Theorem 1, and

$$J_1(t) = k_1^2 \sum_{i=1}^{N} \sum_{j=0}^{N} a_{ij} \left|b_{1ji}(t-\tau_1)\right|^2 + k_2^2 \sum_{i=1}^{N} \sum_{j=0}^{N} a_{ij} \left|b_{2ji}(t-\tau_2)\right|^2.$$

From the definition of $\overline{\delta}(t)$, Hypothesis 2, and $2\delta_y^T(t)(\mathcal{L} \otimes I_n)\delta_y(t) = \sum_{i=1}^{N} \sum_{i=1}^{N} a_{ij}|\delta_{yj}(t) - \delta_{yi}(t)|^2$, we obtain

$$\begin{aligned}
J_1(t) &\le k_1^2\bar{\sigma}_1^2\delta_y^T(t-\tau_1)\left[(2\mathcal{L}+D_0) \otimes I_n\right]\delta_y(t-\tau_1) + k_2^2\bar{\sigma}_2^2\delta_y^T(t-\tau_2)\left[(2\mathcal{L}+D_0) \otimes I_n\right]\delta_y(t-\tau_2) \\
&\le 2k_1^2\bar{\sigma}_1^2\overline{\delta}_y^T(t-\tau_1)(\Lambda_0) \otimes I_n)\overline{\delta}_y(t-\tau_1) + 2k_2^2\bar{\sigma}_2^2\overline{\delta}_y^T(t-\tau_2)(\Lambda_0 \otimes I_n)\overline{\delta}_y(t-\tau_2).
\end{aligned} \tag{46}$$

By the definition of $J_2(t)$, we have

$$J_2(t) = 2\theta\overline{\delta}_y^T(t)(\Phi^T \otimes I_n)\bar{F}(t) + 2\overline{\delta}_v^T(t)(\Phi^T \otimes I_n)\bar{F}(t).$$

It can be obtained that $\Phi\Phi^T = I_N$, $|\delta_y(t)|^2 = |\overline{\delta}_y(t)|^2$, and $|\delta_v(t)|^2 = |\overline{\delta}_v(t)|^2$. Hence, by Hypothesis 3, we get

$$\begin{aligned}
2\overline{\delta}_y^T(t)(\Phi^T \otimes I_n)\bar{F}(t) &= 2\delta_y^T(t)(\Phi\Phi^T \otimes I_n)\bar{F}(t) \\
&= 2\sum_{i=1}^{N} \delta_{yi}^T(t) \left(f\left(y_i(t-\tau_y), v_i(t-\tau_v)\right) - f\left(y_0(t-\tau_y), v_0(t-\tau_v)\right) \right) \\
&\le 2\sum_{i=1}^{N} \left|\delta_{yi}(t)\right| \left(q_1 \left|\delta_{yi}(t-\tau_y)\right| + q_2 \left|\delta_{vi}(t-\tau_v)\right| \right) \\
&\le (q_1+q_2)\left|\overline{\delta}_y(t)\right|^2 + q_1\left|\overline{\delta}_y(t-\tau_y)\right|^2 + q_2\left|\overline{\delta}_v(t-\tau_v)\right|^2.
\end{aligned}$$

Similarly, we have

$$2\overline{\delta}_v^T(t)(\Phi^T \otimes I_n)\bar{F}(t) \le (q_1+q_2)\left|\overline{\delta}_v(t)\right|^2 + q_1\left|\overline{\delta}_y(t-\tau_y)\right|^2 + q_2\left|\overline{\delta}_v(t-\tau_v)\right|^2.$$

The two inequalities aforementioned produce

$$J_2(t) \le (q_1+q_2)\left(\theta\left|\overline{\delta}_y(t)\right|^2 + \left|\overline{\delta}_v(t)\right|^2\right) + (\theta+1)\left[q_1\left|\overline{\delta}_y(t-\tau_y)\right|^2 + q_2\left|\overline{\delta}_v(t-\tau_v)\right|^2\right].$$

The estimations of $J_3(t)$ and $J_4(t)$ have the form (14) and (15), respectively. Let $p_{1i}(t) = f(y_i(t-\tau_y), v_i(t-\tau_v)) - f(y_0(t-\tau_y), v_i(t-\tau_v))$, and $p_{2i}(t) = f(y_0(t-\tau_y), v_i(t-\tau_v)) - f(y_0(t-\tau_y), v_0(t-\tau_v))$. Then, $\bar{F}(t) = p_1(t) + p_2(t)$, where $p_l(t) = [p_{l1}(t)^T, \dots, p_{lN}^T(t)]^T$, $l = 1, 2$. Note that $\overline{\delta}_y = \Phi^T \delta_y$ and $\Lambda_0 = \Phi^T \mathcal{L}_0 \Phi$. Then, we obtain

$$\begin{aligned}
J_5(t) &= -2k_1 \left(\int_{t-\tau_1}^{t} \overline{\delta}_y(s) \mathrm{d}s \right)^T \left(\phi^T \mathcal{L}_0^T \phi \phi^T \otimes I_n \right) \bar{F}(t) \\
&= -2k_1 \left(\int_{t-\tau_1}^{t} \delta_y(s) \mathrm{d}s \right)^T \left(\mathcal{L}_0^T \otimes I_n \right) \left[p_1(t) + p_2(t) \right] \\
&\le 2k_1^2 (q_1 + q_2) \int_{t-\tau_1}^{t} \overline{\delta}_y^T(s) \left(\Lambda_0^2 \otimes I_n \right) \overline{\delta}_y(s) |^2 \mathrm{d}s + \left(\frac{1}{q_2} |p_1(t)|^2 + \frac{1}{q_1} |p_2(t)|^2 \right) \tau_1 \\
&\le k_1^2 (q_1 + q_2) \int_{t-\tau_1}^{t} \overline{\delta}_y^T(s) \left(\Lambda_0^2 \otimes I_n \right) \overline{\delta}_y(s) \mathrm{d}s + q_1 \tau_1 \left| \overline{\delta}_y(t-\tau_y) \right|^2 + q_2 \tau_1 \left| \overline{\delta}_v(t-\tau_v) \right|^2,
\end{aligned}$$

and similarly,

$$J_6(t) \le k_2^2 (q_1 + q_2) \int_{t-\tau_2}^{t} \overline{\delta}_v^T(s) \left(\Lambda_0^2 \otimes I_n \right) \overline{\delta}_v(s) \mathrm{d}s + q_1 \tau_2 \left| \overline{\delta}_y(t-\tau_y) \right|^2 + q_2 \tau_2 \left| \overline{\delta}_v(t-\tau_v) \right|^2.$$

Hence, we get

$$\begin{aligned}
\mathfrak{L} V_1(\overline{\delta}_t) \le{}& S_1 \left(\overline{\delta}(t) \right) + \int_{t-\tau_1}^{t} \overline{\delta}^T(s) \left(L_1^T P L_1 \otimes I_n \right) \overline{\delta}(s) \mathrm{d}s + \int_{t-\tau_2}^{t} \overline{\delta}^T(s) \left(L_2^T P L_2 \otimes I_n \right) \overline{\delta}(s) \mathrm{d}s \\
&+ (\theta + 1) \left[q_1 \left| \overline{\delta}_y(t-\tau_y) \right|^2 + q_2 \left| \overline{\delta}_v(t-\tau_v) \right|^2 \right] + 2k_1^2 \sigma_1^2 \overline{\delta}_y^T(t-\tau_1) [\Lambda_0 \otimes I_n] \overline{\delta}_y(t-\tau_1) \\
&+ k_1^2 (q_1 + q_2) \int_{t-\tau_1}^{t} \overline{\delta}_y^T(s) \left(\Lambda_0^2 \otimes I_n \right) \overline{\delta}_y(s) \mathrm{d}s + 2k_2^2 \sigma_2^2 \overline{\delta}_v^T(t-\tau_2) [\Lambda_0 \otimes I_n] \overline{\delta}_v(t-\tau_2) \\
&+ k_2^2 (q_1 + q_2) \int_{t-\tau_2}^{t} \overline{\delta}_v^T(s) \left(\Lambda_0^2 \otimes I_n \right) \overline{\delta}_v(s) \mathrm{d}s \\
&+ (\tau_1 + \tau_2) \left(q_1 \left| \overline{\delta}_y(t-\tau_y) \right|^2 + q_2 \left| \overline{\delta}_v(t-\tau_v) \right|^2 \right),
\end{aligned}$$

where

$$S_1 \left(\overline{\delta}(t) \right) = \overline{\delta}^T(t) \left(\left(L^T P + PL + L^T PL(\tau_1 + \tau_2) \right) \otimes I_n \right) \overline{\delta}(t) + (q_1 + q_2) \left(\theta \left| \overline{\delta}_y(t) \right|^2 + \left| \overline{\delta}_v(t) \right|^2 \right).$$

Hence, from the definition of $V(\cdot)$, we obtain (18) with $Q = L^T P + PL + L^T PL(\tau_1 + \tau_2) + L_1^T P L_1 \tau_1 + L_2^T P L_2 \tau_2 + D$, $D = \begin{bmatrix} d_{11}(\theta) & 0 \\ 0 & d_{22}(\theta) \end{bmatrix}$, $d_{11}(\theta) = (2q_1 + q_2)\theta I_{N-1} + (\tau_1 + \tau_2 + 1) q_1 I_{N-1} + (q_1 + q_2) k_1^2 \Lambda_0^2 \tau_1 + 2k_1^2 \sigma_1^2 \Lambda_0$, and $d_{22}(\theta) = (q_2 \theta + 2q_2 + q_1 + q_2(\tau_1 + \tau_2)) I_{N-1} + (q_1 + q_2) k_2^2 \Lambda_0^2 \tau_2 + 2k_2^2 \sigma_2^2 \Lambda_0$.

Thus, following the proof of Theorem 1, we can obtain the desired assertions. □

Remark 9. If the graph $\widetilde{\mathcal{G}}$ forms a star graph, then $\mathcal{L} = 0$, $D_0 = \Lambda_0 = I_N$, $\Phi = I_N$, and (46) can be estimated by $J_1(t) \le k_1^2 \bar{\sigma}_1^2 \|\delta_y(t-\tau_1)\|^2 + k_2^2 \bar{\sigma}_2^2 \|\delta_y(t-\tau_2)\|^2$. In this case, (43) and (44) with $\lambda_{01} = 1$ and $\bar{\sigma}_1^2, \bar{\sigma}_2^2$ being replaced by $0.5\bar{\sigma}_1^2, 0.5\bar{\sigma}_2^2$ are the corresponding consensus conditions.

4 | SIMULATIONS

We consider mean square and almost sure consensus for nonlinear and linear scalar four-agent systems with the topology graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, 3, 4\}$, $\mathcal{E} = \{(1,2), (2,3), (3,4), (4,3), (3,2), (2,1)\}$ and $\mathcal{A} = [a_{ij}]_{4\times 4}$ with $a_{21} = a_{12} = a_{32} = a_{23} = a_{43} = a_{43} = 1$ and other being zero. It is easy to see that the graph $\mathcal{G}$ is strongly connected. Moreover, we can compute the eigenvalues of the corresponding Laplacian matrix $\mathcal{L}$, ie, $\lambda_1 = 0$, $\lambda_2 = 0.5858$, $\lambda_3 = 2$, and $\lambda_4 = 3.4142$. We first investigate the nonlinear case with delays in the intrinsic dynamics and measurements.

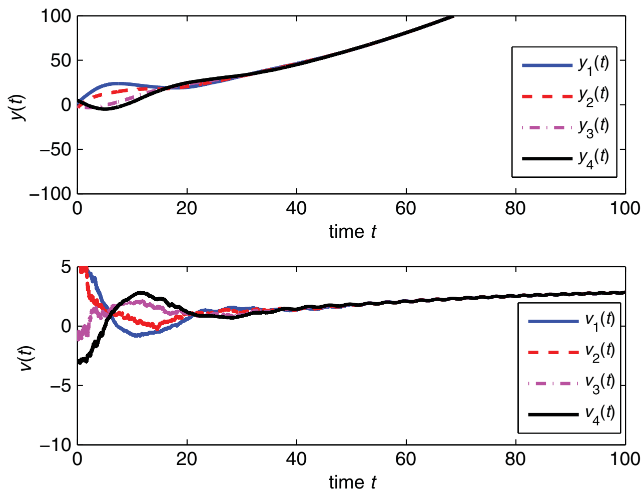

FIGURE 1 States of the four nonlinear agents for one random path [Colour figure can be viewed at wileyonlinelibrary.com]

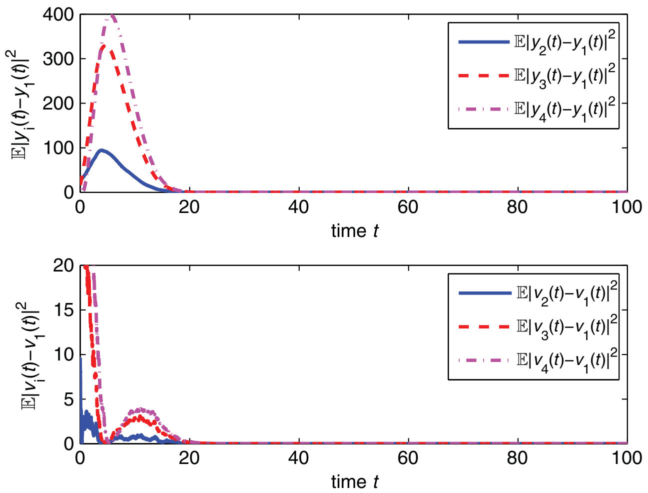

FIGURE 2 Mean square errors of the relative states for the case with nonlinear dynamic [Colour figure can be viewed at wileyonlinelibrary.com]

Nonlinear case with delays: $f(y, v, t) = 0.001 \sin(y) + 0.01 \cos(v)$. Assume the delays in the intrinsic dynamics $\tau_y = 0.8$, $\tau_v = 1$, and the delays in the measurements $\tau_1 = 0.1$ and $\tau_2 = 0.2$. The noise intensity functions in the position measurement and velocity measurement have the forms $g_{1ji}(y) = \bar{\sigma}_1 y$, $g_{2ji}(v) = \bar{\sigma}_2 v$, for any $y, v \in \mathbb{R}$. We assume that $\bar{\sigma}_1 = 0.2$ and $\bar{\sigma}_2 = 0.4$. The initial values are given as $y(t) = [2, -4, -2, 5]^T$, $v(t) = [5, 8, -1, -3]^T$, for all $t \in [-\tau, 0]$, where $\tau = \tau_y \vee \tau_v \vee \tau_1 \vee \tau_2$.

It is easy to see that Hypothesis 3 holds with $q_1 = 0.001$ and $q_2 = 0.01$. In order for mean square and almost sure consensus, we first choose $0.52 = k_2 < \frac{2}{2\frac{N-1}{N}\bar{\sigma}_2^2 + [(q_1+q_2+2)\tau_2+\tau_1]\lambda_N} = 1.0232$, which will guarantee $\varrho(k_2) = 0.2996 > 0$. Then, we choose $k_1 = 0.075$. It is easy to verify $2k_1\lambda_2 > 2q_1 + q_2$ and to compute $\rho(k_1) = 0.2887 < \varrho(k_2) = 0.2996$. Hence, from Theorem 1, we know that the second-order system can achieve mean square and almost sure consensus.

To see almost sure consensus, we take one random path and have Figure 1, which shows that the positions and velocities of the four agents tend to get together, respectively. That is, almost sure consensus is revealed. In order to simulate mean square consensus, we consider the behaviors of the relative states $\{|y_i(t) - y_1(t)|\}_{i=2,3,4}$ and $\{|v_i(t) - v_1(t)|\}_{i=2,3,4}$. We generate $p = 10^4$ sample paths, and relative states $\{|y_i(t) - y_1(t)|\}_{i=2,3,4}$ and $\{|v_i(t) - v_1(t)|\}_{i=2,3,4}$ under the pth path are denoted as $\{|y_i^p(t) - y_1^p(t)|\}_{i=2,3,4}$ and $\{|v_i^p(t) - v_1^p(t)|\}_{i=2,3,4}$. Then, we take mean square average, ie, we use $\frac{1}{10^4}\sum_{p=1}^{10^4} |y_i^p(t) - y_1^p(t)|^2$ to approximate $\mathbb{E}|y_i^p(t) - y_1^p(t)|^2$, and obtain Figure 2, which shows that the four agents achieve mean square consensus.

Linear case without delay: $f(y, v, t) = q_1 y + q_2 v$. This simulation is to show the effectiveness of Theorem 2 under $\tau_y = \tau_v = \tau_1 = \tau_2 = 0$. We first see that conditions (20) and (21) can guarantee mean square and almost sure consensus.

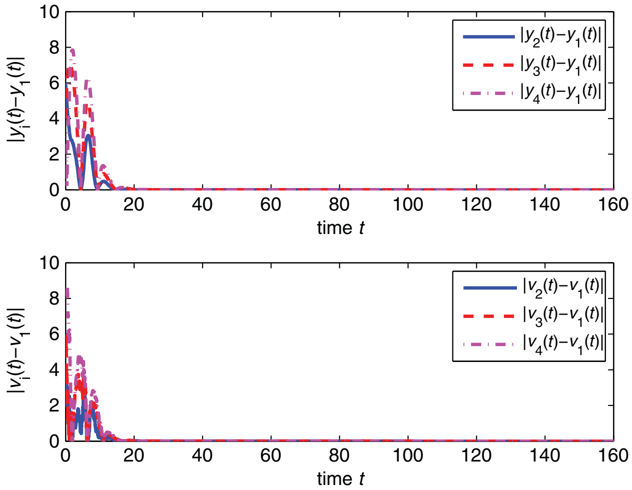

FIGURE 3 Errors of the relative states for the case with linear dynamics: $k_1 = 1, k_2 = 0.8$, and $\sigma_2 = 0.4$ [Colour figure can be viewed at wileyonlinelibrary.com]

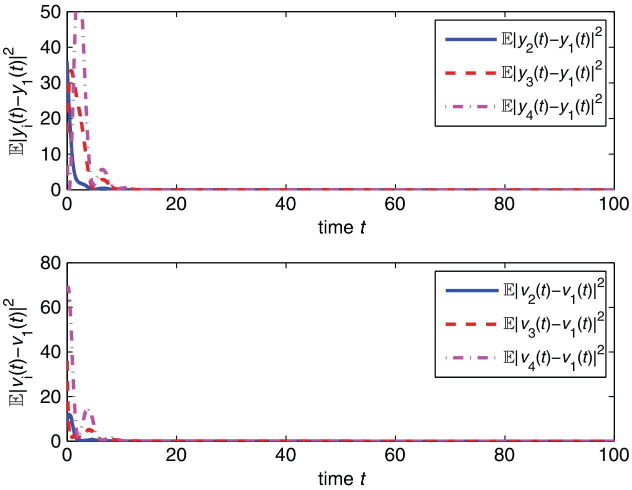

FIGURE 4 Mean square errors of the relative states for the case with linear dynamics: $k_1 = 1, k_2 = 0.8$, and $\sigma_2 = 0.4$ [Colour figure can be viewed at wileyonlinelibrary.com]

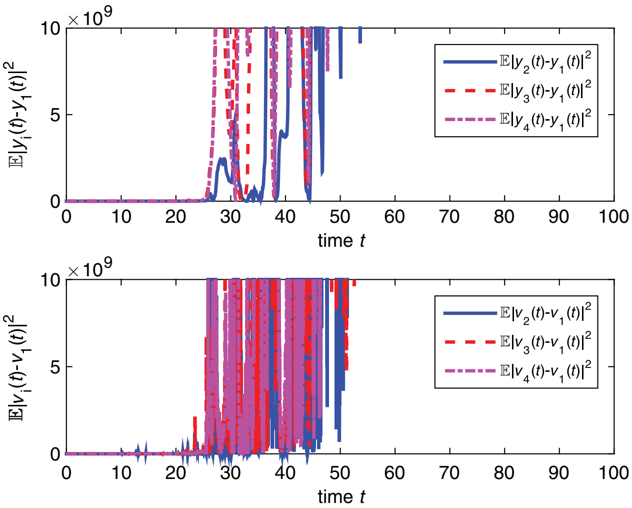

FIGURE 5 Mean square errors of the relative states for the case with linear dynamics: $k_1 = 1, k_2 = 0.8$, and $\sigma_2 = 2$ [Colour figure can be viewed at wileyonlinelibrary.com]

Assume that $q_1 = 0.2, q_2 = 0.3$, and choose $k_1 = 1, k_2 = 0.8$. Then, it is easy to see $0.0456 = \frac{k_1^2\sigma_1^2\frac{N-1}{N}}{k_1-\frac{q_1}{\lambda_2}} < k_2\lambda_2 - q_2 - \frac{N-1}{N}k_2^2\sigma_2^2\lambda_2 = 0.1237$. Theorem 2 yields that almost sure and mean square consensus can be achieved. Simulations in Figures 3 and 4 confirm the sufficiency of the condition (21).

If the noise intensity $\sigma_2 = 2$, then $4q_2\frac{N-1}{N}\sigma_2^2 - \lambda_N = 0.1858 > 0$ and the condition (22) is defied for any k_1, k_2 since $Q(k_1, k_2) := k_2\lambda_N - q_2 - \frac{N-1}{N}k_2^2\sigma_2^2\lambda_N < 0$ for any k_1, k_2. From Theorem 2, we know that mean square consensus cannot be achieved for any choice k_1, k_2. For the choice $k_1 = 1, k_2 = 0.8$, $Q(k_1, k_2) = -4.1239$, and we can obtain Figure 5, which shows that mean square consensus cannot be achieved without condition (22).

5 | CONCLUSION

In this paper, the consensus conditions of the delayed multiagent systems with multiplicative noises have been derived by using degenerate Lyapunov functionals and stochastic stability theorem. The necessary conditions and sufficient conditions are established for nonlinear and linear multiagent systems. For the case with the coexistence of the intrinsic dynamics and multiplicative noises, it is proved that if the noise intensities and the Lipschitz constant of the intrinsic dynamics are small enough, then multiagent systems can achieve mean square and almost sure consensus. However, for the delay-free case with linear dynamics, it is proved that, if noise intensities and Lipschitz constant are large enough, then multiagent systems may not be mean square consentable with respect to weighted average type control protocols, ie, mean square consensus protocol based on the relative state measurements may not exist. We also extend the leader-free multiagent systems to the leader-following case and obtain the sufficient conditions for mean square and almost sure consensus and stabilization.

It is worthy noting that the case with nonidentical measurement delays has been investigated in the works of Bliman and Ferrari-Trecate[46] and Münz et al[47] for deterministic systems, but it is still difficult to extend this to stochastic systems. We hope to discuss it in the future work. Moreover, in this paper, we assume that the topology is undirected and fixed. We hope to further the current work to the cases with the general directed and switching graphs. Another future research topic is stochastic consensus based on the event-trigger mechanism.

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ORCID

Tao Li http://orcid.org/0000-0001-6173-7987

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